Unit

4

Partial Differential Equations

Partial Differential Equations

A partial differential equation is an equation in which x and y represent the independent variables and z the dependent variable so that $z = f(x, y)$.

We also use the notations $\frac{\partial Z}{\partial x} = p,\ \frac{\partial Z}{\partial y} = q,\ \frac{\partial^2 Z}{\partial x^2} = r,\ \frac{\partial^2 Z}{\partial x \partial y} = s,\ \frac{\partial^2 Z}{\partial y^2} = t.$

Order of a Partial Differential Equation

The order of a partial differential equation is the order of the highest partial derivative occurring in the equation and its degree is the degree of their derivative.

For example, (1) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

Here, z is dependent and x, y are independent and this equation is of order one and degree one.

(2) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 Z}{\partial^2 y} + \frac{\partial^2 v}{\partial z^2} = 0$

Here, v is dependent and x, y, z are independent and this equation is of order two and degree one.

Partial differential equations will play an important role in the study of wave equation, heat equation, electromagnetism, radar, ratio, television and other fields.

Formation of Partial Differential Equation

Partial differential equation can be obtained by:

(i) Elimination of arbitrary constants

(ii) Elimination of arbitrary functions involving two or more variables.

(i) Elimination of Arbitrary Constants

Let $f(x, y, z, a, b) = 0$ ...(1)

be an equation involving two arbitrary constants $'a'$ and $'b'$. Differentiating this equation partially with respect to x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial x}\right) = 0 \quad ...(2)$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial y}\right)=0 \qquad \ldots(3)$$

By eliminating a, b from equation (1), (2) and (3), we get an equation of the form

$$f(x,y,z,p,q)=0 \qquad \ldots(4)$$

which is a partial differential equation of first order.

Example Problems

Example 1: By eliminating the constant, obtain the partial differential equation from the relation $2z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$.

Solution: Given $2z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ ...(1)

Differentiating equation (1) with respect to x,

$$2\frac{\partial z}{\partial x}=\frac{2x}{a^2}$$

$$\frac{\partial z}{\partial x}=\frac{x}{a^2}$$

$$p=\frac{x}{a^2}$$

$$\frac{p}{x}=\frac{1}{a^2}$$

Differentiating equation (1) with respect to y,

$$2\frac{\partial z}{\partial y}=\frac{2y}{b^2}$$

$$\frac{\partial z}{\partial y}=\frac{y}{b^2}$$

$$q=\frac{y}{b^2}$$

$$\frac{q}{y}=\frac{1}{b^2}$$

Substituting $\frac{1}{a^2}, \frac{1}{b^2}$ in equation (1), we get the required partial differentiation equation

$$2z=x^2\frac{p}{x}+y^2\frac{q}{y}$$

$$2z=px+qy$$

Example 2: Form the partial differential equations, by eliminating the arbitrary constants from $z=(x^2+a)(y^2+b)$.

Solution: Given equation is $z=(x^2+a)(y^2+b)$ ...(1)

Differentiating equation (1) with respect to x, we get

$$\frac{\partial z}{\partial x}=2x(y^2+b)$$

$$p=2x(y^2+b)$$

$$\frac{p}{2x}=(y^2+b) \qquad \ldots(2)$$

Differentiating equation (1) with respect to y, we get

$$\frac{\partial z}{\partial y}=2y(x^2+a)$$

$q = 2y(x^2 + a)$

$\frac{q}{2y} = (x^2 + a)$ ...(3)

From (2) and (3), we obtain,

$z = \frac{q}{2x} \cdot \frac{p}{2y}$

$4xyz = pq$

which is the required differential equation.

Example 3: Find the differential equations of all planes which are at a constant distance r from the origin.

Solution: Equation of all the planes which are at a constant distance r from origin is:

$ax + by + cz = r$ ...(1)

with $a^2 + b^2 + c^2 = 1$ ...(2)

From (2), we have

$c = \sqrt{1 - a^2 - b^2}$

Differentiating (1) partially with respect to x, we get

$a + c\frac{\partial z}{\partial x} = 0 \quad \left(\because \frac{\partial z}{\partial x} = p = \frac{-a}{c}\right), a + cp = 0$

$a = -pc$ ...(3)

Differentiating (1) partially with respect to y, we get

$b + c\frac{\partial z}{\partial y} = 0 \left(\frac{\partial z}{\partial y} = q = \frac{-b}{c}\right), b + cq = 0$

$b = -qc$ ...(4)

Substituting a and b in (2), we get

$p^2c^2 + q^2c^2 + c^2 = 1$

$\frac{1}{c} = \sqrt{p^2 + q^2 + 1}$ ...(5)

From (1), we have

$z = \frac{r}{c} - \frac{a}{c}x - \frac{b}{c}y$

From (3), (4) and (5), we have

$z = px + qy + r\sqrt{p^2 + q^2 + 1}$

Example 4: Form the partial differentiation equation from $z = ax^2 + by^2 + ab$, by eliminating arbitrary constants a and b.

Solution: Differentiating $z = ax^2 + by^2 + ab$ ...(1)

partially with respect to x, we get

$\frac{\partial z}{\partial x} = 2ax \Rightarrow p = 2ax \Rightarrow \frac{p}{2x} = a$

Differentiating (1) partially with respect to y, we get

$\frac{\partial z}{\partial y} = 2by \Rightarrow q = 2by \Rightarrow \frac{q}{2y} = b$

Substituting a and b in equation (1), we get

$$z = \left(\frac{p}{2x}\right)x^2 + \left(\frac{q}{2y}\right)y^2 + \frac{p}{2x}\cdot\frac{q}{2y}$$

$$4xyz = pq + 2px^2y + 2qy^2x$$

This is the required partial differential.

Example 5: Form a partial differential equation by eliminating a, b and c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution: Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ...(1)

Differentiating (1) partially with respect to x, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\cdot\frac{\partial z}{\partial x} = 0$$

$$c^2x + a^2z\frac{\partial z}{\partial x} = 0 \quad \ldots(2)$$

Differentiating (1) partially with respect to y, we get

$$\frac{2y}{b^2} + \frac{2z}{c^2}\cdot\frac{\partial z}{\partial y} = 0$$

$$c^2y + b^2z\frac{\partial z}{\partial y} = 0 \quad \ldots(3)$$

Since there are three constants, again differentiating (2) with respect to x, we get

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0$$

$$c^2 = -a^2(p^2 + zr)$$

Substituting c^2 in equation (2), we get

$$-a^2(p^2 + zr)x + a^2zp = 0$$

$$-a^2(xp^2 + xzr - zp) = 0$$

$$xp^2 + xzr - zp = 0$$

which is second order partial differential equation.

Note: We can differentiate (3) with respect to y also so that we get $yq^2 + yzr - zq = 0$. So, we understand that more partial differential equations for a given relation between variables.

Example 6: Form a partial differential equation by eliminating the arbitrary constants h and k from $(x-h)^2 + (y-k)^2 + z^2 = c^2$.

(or)

Find the partial differential equation of a family of sphere with centre in xy-plane and having radius c.

Solution: The equation of a family of sphere with centre in xy-plane and having radius c is $(x-h)^2 + (y-k)^2 + z^2 = c^2$

Differentiating above equation with respect to x and y, we get

$$2(x-h) + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow x - h = -zp$$

and $2(y-k) + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow y - k = -zq$

Substituting $x-h$ and $y-k$ in given relation, we get

$$z^2p^2 + z^2q^2 + z^2 = c^2 \Rightarrow z^2(p^2 + q^2 + 1) = c^2$$

which is the required partial differential equation.

Example 7: Form the differential equations of all planes whose x-intercept is always equal to the y-intercept.

Solution: Equation of the planes is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ...(1)

Since x and y-intercept are equal,

$\therefore$ Equation of such planar is $\frac{x+y}{a}+\frac{z}{c}=1$ ...(2)

Differentiating (2) with respect to x, we get

$\frac{1+0}{a}+\frac{1}{c}\frac{\partial z}{\partial x}=0$

$\frac{1}{a}+\frac{p}{c}=0$ ...(3)

Differentiating (2) with respect to y, we get

$\frac{1+0}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0$

$\frac{1}{a}+\frac{q}{c}=0$ ...(4)

From (3), we have

$\frac{1}{a}=-\frac{1}{c}p$

From (4), we have

$\frac{1}{a}=-\frac{1}{c}q$

Dividing (3) by (4), we get

$1=\frac{p}{q}\Rightarrow p=q$ is required equation.

Example 8: Eliminate a and b from $z=\frac{x}{a}+\frac{y}{b}$.

Solution: Given $z=\frac{x}{a}+\frac{y}{b}$ ...(1)

Partially differentiating equation (1) with respect to x, we get

$a=\frac{1}{p}$ ...(2)

Partially differentiating equation (1) with respect to y, we get

$b=\frac{1}{q}$ ...(3)

Substituting a and b values in equation (1), we get

$z=px+qy$

Example 9: Form the differential equation from $2z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ by eliminating arbitrary constants a and b.

Solution: Given $2z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ ...(1)

Differentiating partially with respect to x, we get

$2\frac{\partial z}{\partial x}=\frac{2x}{a^2}$

$\frac{\partial z}{\partial x} = \frac{x}{a^2}$

$p = \frac{x}{a^2}$

$\frac{p}{x} = \frac{1}{a^2}$

Differentiating equation (1) with respect to y, we get

$2\frac{\partial z}{\partial y} = \frac{2y}{b^2}$

$\frac{\partial z}{\partial y} = \frac{y}{b^2}$

$q = \frac{y}{b^2}$

$\frac{q}{y} = \frac{1}{b^2}$

Substituting $\frac{1}{a^2}$ and $\frac{1}{b^2}$ in equation (1), we get the required partial differentiation equation

$2z = x^2\frac{p}{x} + y^2\frac{q}{y}$

$2z = px + qy$

Example 10: Find the partial differential equation of the family of sphere of radius 5 with centres on the plane $x = y$.

Solution: The equation of the family of the sphere is $(x-a)^2 + (y-a)^2 + (z-b)^2 = 25$, where a and b are arbitrary constants.

Differentiating above equation with respect to x and y, we get

$(x-a) + (z-b)p = 0$

and $(y-a) + (z-b)q = 0$

Substituting $x - a = -p(z-b)$, and $y - a = -q(z-b)$ in the family of sphere equation, we get

$p^2(z-b)^2 + q^2(z-b)^2 + (z-b)^2 = 25$

$(z-b)^2(p^2 + q^2 + 1) = 25$

$x - y = (x-a) - (y-a) = (q-p)(z-b)$

$z - b = \frac{x-y}{q-p}$

Substituting $z - b$ in the above equation, we get

$(x-y)^2(p^2 + q^2 + 1) = 25(q-p)^2$

which is required partial differential equation.

Exercise:

(1) Form the partial differential equation from the following relations between variables by eliminating arbitrary constants.

(i) $ax^2 + by^2 + z^2 = 1$

(ii) $z = ax + by + ab$

(iii) $\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1$

(2) Find the differential equation of the sphere whose centre lies on the z-axis.

Answers:

(1) (i) $z(px+qy)=z^2-1$

(ii) $z=px+qy+pq$

(iii) $py=qx$

(2) $py-qx=0$

Hint: The equation of the family of sphere having their centre on z-axis and having radius r is $(x-0)^2+(y-0)^2+(z-c)^2=r^2$, i.e., $x^2+y^2+(z-c)^2=r^2$, where c and r are arbitrary constants.

(ii) Formation of Partial Differential Equation by Eliminating Arbitrary Functions

In this, we have three cases:

Case I: Eliminating one arbitrary function from $z=f(u)$

Consider $z=f(u)$ ...(1)

where $f(u)$ is an arbitrary function of u and v is known function of x, y and z.

Differentiating equation (1) partially with respect to x and y by chain rule, we get

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x} \quad \text{...(2)}$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y} \quad \text{...(3)}$$

By eliminating the arbitrary function f from (1), (2) and (3), we get the partial differential equation of first order.

Case II: Eliminating one arbitrary function from $f(u,v)=0$

Let u and v are the two functions of x, y and z connected by the relation

$f(u,v)=0$ ...(1)

Differentiating (1) with respect to x and y, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0$$

and $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=0$

=> $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\cdot p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p\right)=0$ ...(2)

and $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot q\right)=0$ ...(3)

where $\frac{\partial z}{\partial x}=p,\ \frac{\partial z}{\partial y}=q$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from equation (2) and (3), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot q & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot q \end{vmatrix}=0$$

=> $\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot q\right)-\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot q\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p\right)=0$

$$=> p\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z}\right)+q\left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

or $Pp + Qq = R$ …(4)

is the required partial differential equation

where, $P=\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z}$, $Q=\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}$, $R=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$

Case III: Eliminating two arbitrary functions

When the given relation contains two arbitrary functions, then differentiate twice or thrice and elimination process becomes a partial differential equation of second and higher order.

Example Problems

Example 1: By eliminating the arbitrary function from $z = e^{ny}\phi(x-y)$, obtain a partial differential equation.

Solution: Differentiating with respect to y, we get

$$\frac{\partial z}{\partial y} = ne^{xy}\phi(x-y) - e^{ny}\phi'(x-y)$$

$$=> q = ne^{xy}\phi(x-y) - e^{ny}\phi'(x-y)$$

$$=> q = nz - p$$

i.e., $p + q - nz = 0$ is the required equation.

Example 2: Form the partial differential equation by eliminating the function from $z = y^2 + 2F\left(\frac{1}{x}+\log y\right)$.

Solution: Differentiating the given relation partially with respect to x, we get

$$\frac{\partial z}{\partial x} = 0 + 2F'\left(\frac{1}{x}+\log y\right)\left(-{}^{1}/_{x^2}\right)$$

$$=> -px^2 = 2F'\left(\frac{1}{x}+\log y\right) \quad \ldots(1)$$

Differentiating the given relation partially with respect to y, we get

$$\frac{\partial z}{\partial y} = 2y + 2F'\left(\frac{1}{x}+\log y\right)\frac{1}{y}$$

Replacing $2F'\left(\frac{1}{x}+\log y\right)$ from (1), we get

$q = 2y - \frac{px^2}{y}$ or $px^2 + qy = 2y^2$

which is required partial differential equation.

Example 3: Form a partial differential equation by eliminating the arbitrary functions,

(i) $z = f(x^2+y^2)$ (ii) $z = xy + f(x^2+y^2)$ (iii) $z = x^n f\left({}^{y}/_{x}\right)$

Solution:

(i) $z = f(x^2+y^2)$ …(1)

We have to eliminate the arbitrary function $'f'$.

Differentiating (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = f'(x^2+y^2)\cdot 2x$$

$$p = f'(x^2+y^2)\cdot 2x \quad \ldots(2)$$

Similarly, differentiating (1) partially with respect to y, we get

$\frac{\partial z}{\partial y} = f'(x^2 + y^2) \cdot 2y$

$=> q = f'(x^2 + y^2)2y$ …(3)

$\therefore$ Dividing (2) by (3), we get

$\frac{p}{q} = \frac{f'(x^2+y^2)2x}{f'(x^2+y^2)2y}$

$=> py - qx = 0$ is required partial differential equation.

(ii) $z = xy + f(x^2 + y^2)$ …(1)

Differentiating (1) partially with respect to x, we get

$\frac{\partial z}{\partial x} = y + f'(x^2 + y^2) \cdot 2x$

$p - y = f'(x^2 + y^2) \cdot 2x$ …(2)

Similarly, differentiating (1) partially with respect to y, we get

$\frac{\partial z}{\partial y} = x + f'(x^2 + y^2) \cdot 2y$

$q - x = f'(x^2 + y^2)2y$ …(3)

(2) ÷ (3) gives

$\frac{p-y}{q-x} = \frac{f'(x^2+y^2)2x}{f'(x^2+y^2)2y}$

$(p - y)y = (q - x)x$

$=> x^2 - y^2 = qx - py$ is required partial differential equation.

(iii) $z = x^n f\left({}^{y}/_{x}\right)$ …(1)

Differentiating (1) partially with respect to x, we get

$\frac{\partial z}{\partial x} = nx^{n-1} f\left({}^{y}/_{x}\right) + x^n f'\left({}^{y}/_{x}\right)\left(-\frac{y}{x^2}\right)$

$p = \frac{nx^n}{x} f\left({}^{y}/_{x}\right) - \frac{x^{n-1} y f'\left({}^{y}/_{x}\right)}{x}$

$p = \frac{nx^n f\left({}^{y}/_{x}\right) - x^{n-1} y f'\left({}^{y}/_{x}\right)}{x}$

$px = nx^n f\left({}^{y}/_{x}\right) - x^{n-1} y f'\left({}^{y}/_{x}\right)$ …(2)

Similarly, differentiating (1) with respect to y, we get

$\frac{\partial z}{\partial y} = x^n f'\left({}^{y}/_{x}\right) \cdot \frac{1}{x}$

$q = x^{n-1} f'\left({}^{y}/_{x}\right)$

Multiplying y on both sides, we get

$qy = x^{n-1} y f'\left({}^{y}/_{x}\right)$ …(3)

Adding (2) and (3), we get

$$px + qy = nx^n f\left({}^{y}/_{x}\right) - x^{n-1} y f'\left({}^{y}/_{x}\right) + x^{n-1} y f'\left({}^{y}/_{x}\right)$$
$$= nx^n f\left({}^{y}/_{x}\right)$$

$\therefore px + qy = nz$ (from (1))

Example 4: Form a partial differential equation by eliminating the arbitrary function $\emptyset$ from $lx + my + nz = \phi(x^2 + y^2 + z^2)$.

Solution: Given $lx + my + nz = \phi(x^2 + y^2 + z^2)$ ...(1)

Differentiating (1) with respect to x, we get

$$l + n\frac{\partial z}{\partial x} = \phi'(x^2 + y^2 + z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right)$$

$$=> l + np = \phi'(x^2 + y^2 + z^2)(2x + 2zp) \quad ...(2)$$

Differentiating (2) with respect to y, we get

$$m + n\frac{\partial z}{\partial y} = \phi'(x^2 + y^2 + z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right)$$

$$m + nq = \phi'(x^2 + y^2 + z^2)(2y + 2zq) \quad ...(3)$$

Dividing (2) by (3), we get

$$\frac{l+np}{m+nq} = \frac{\phi'(x^2+y^2+z^2)(2x+2zp)}{\phi'(x^2+y^2+z^2)(2y+2zq)}$$

$$(l + np)(y + zq) = (m + nq)(x + zp)$$

or $x(m + nq) - (1 + np)y - (lq - mp)z = 0$

which is required partial differential equation.

Example 5: Find the differential equation arising from $\phi(x + y + z, x^2 + y^2 + z^2) = 0$.

Solution: Let $x + y + z = u,\ x^2 + y^2 + z^2 = v$ so that the given relation is

$$\phi(u, v) = 0 \quad ...(1)$$

Differentiating (1) partially with respect to x by chain rule, we get

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial u}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0$$

i.e., $\frac{\partial \phi}{\partial u}\left(1 + \frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(2x + 2z\frac{\partial z}{\partial x}\right) = 0$

$$=> \frac{\partial \phi}{\partial u}(1 + p) + \frac{\partial \phi}{\partial v}(2x + 2zp) = 0 \quad ...(2)$$

Similarly, differentiating with respect to y, we get

$$\frac{\partial \phi}{\partial u}(1 + q) + \frac{\partial \phi}{\partial v}(2x + 2zq) = 0 \quad ...(3)$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ between (2) and (3), we get

$$\begin{vmatrix} 1 + p & 2(x + 2p) \\ 1 + q & 2(y + zq) \end{vmatrix} = 0$$

$$=> (y - z)p + (z - x)q = x - y$$

which is partial differential equation.

Example 6: Form the partial differential equation by eliminating the arbitrary functions from $\phi(z - xy, x^2 + y^2) = 0$.

Solution: Given $\phi(z - xy, x^2 + y^2) = 0$

Let $u = z - xy, v = x^2 + y^2$ so that the given relation is $\phi(u, v) = 0$.

Differentiating with respect to x and y, we get

$$\frac{\partial\phi}{\partial u}(p-y)+\frac{\partial\phi}{\partial v}(2x)=0 \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial u}(q-x)+\frac{\partial\phi}{\partial v}(2y)=0 \qquad \ldots(2)$$

Eliminating $\frac{\partial\phi}{\partial u},\frac{\partial\phi}{\partial v}$ from above equation, we get

$$\begin{vmatrix} p-y & 2x \\ q-x & 2y \end{vmatrix}=0$$

$$=> 2py-2y^2-2qx+2x^2=0$$

$$=> py-y^2-qx+x^2=0$$

$$py-qx=y^2-x^2$$

which is required partial differential equation of the form $Pp+Qq=R$.

Example 7: Form the partial differential equation by eliminating the arbitrary functions from $\phi(x^2+y^2,\ z-xy)=0$.

Solution: Given $\phi(x^2+y^2,\ z-xy)=0$

Let $u=x^2+y^2,\ v=z-xy$ so that the given relation is $\phi(u,v)=0$ …(1)

Differentiate (1) with respect to x and y, we get

$$\frac{\partial\phi}{\partial u}(2x)+\frac{\partial\phi}{\partial v}(p-y)=0 \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial u}(2y)+\frac{\partial\phi}{\partial v}(q-x)=0 \qquad \ldots(3)$$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from (2) and (3), we get

$$\begin{vmatrix} 2x & p-y \\ 2y & q-x \end{vmatrix}=0$$

$$2x(q-x)-2y(p-y)=0$$

$$qx-x^2-py+y^2=0$$

$$-py+qx=x^2-y^2$$

$$py-qx=y^2-x^2$$

which is required partial differential equation.

Example 8: Form the partial differential equation by eliminating the arbitrary functions from $z=e^{ax+by}f(ax-by)$.

Solution: Given $z=e^{ax+by}f(ax-by)$ …(1)

Differentiate (1) with respect to x and y, we get

$$\frac{\partial z}{\partial x}=e^{ax+by}af(ax+by)+e^{ax+by}f'(ax+by)\cdot a$$

$$p=ae^{ax+by}f(ax+by)+ae^{ax+by}f'(ax+by) \qquad \ldots(2)$$

Similarly, $q=be^{ax+by}f(ax+by)-be^{ax+by}f'(ax+by)$ …(3)

Multiplying (2) with b and (3) with a and add, we get

$$bp+aq=2abe^{ax+by}f(ax-by)$$

$$=2abz \qquad (\because z=e^{ax+by}f(ax-by))$$

$\therefore bp + aq = 2abz$

which is required partial differential equation.

Exercise:

(1) Form a partial differential equation by eliminating the arbitrary functions from:

(i) $z = f(x) + e^y g(x)$

(ii) $z = f(x^2 - y^2)$

(2) Form the partial differential equation by eliminating the arbitrary functions from:

(i) $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$

(ii) $\phi\left(\frac{z}{x^3}, \frac{y}{x}\right)$

Answers:

(1) (i) $t - q = 0$

(ii) $yp + xq = 0$

(2) (i) $z(q - p) + y - x = 0$

(ii) $px + qy = 3z$

Equations Easily Integrable

Some of the partial differential equations can be formed by direct integration.

Solutions of a Partial Differential Equation

We understood that a partial differential equation can be formed by eliminating arbitrary constants or arbitrary functions from an equation which involves two or more independent variables.

Linear and Non-linear Partial Differential Equations

Consider a partial differential equation of the form,

$$F(x, y, z, p, q) = 0 \qquad \ldots(1)$$

In this, if p and q are linear, then the given partial difeerential equation is called linear partial differential equation and if it is not linear in p and q, then it is non-linear partial differential equation.

Complete Integral or Complete Solution

A relation of the type $f(x,\ y,\ z,\ a,\ b) = 0$ from which by eliminating a and b, we get $F(x,\ y,\ z,\ p,\ q) = 0$ is called complete integral or complete solution of the partial differential equation $F(x,\ y,\ z,\ p,\ q) = 0$.

Particular Solution

A solution of $F(x,\ y,\ z,\ p,\ q) = 0$ obtained by giving particular values to a and b in the complete integral $f(x,\ y,\ z,\ a,\ b) = 0$ is called a particular integral.

Example Problems

Example 1: Solve $\frac{\partial^2 y}{\partial x^2} = \sin y$.

Solution: Given $\frac{\partial^2 y}{\partial x^2} = \sin y \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \sin y$

Integrating with respect to x treating y as constant and keeping $f(y)$ as the constant of integration, we get

$$\frac{\partial z}{\partial x} = x \sin y + f(y)$$

In the same way, integrating with respect to x treating y as constant and keeping $g(y)$ as the constant of integration, we get

$z = \frac{x^2}{2} \sin y + x f(y) + g(y)$ is the solution.

Example 2: Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$.

Solution: Keeping y fixed and integrating the given equation twice, we get

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2 y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

and $\frac{\partial z}{\partial y} + 3x^3 y^2 - \frac{1}{4}\sin(2x - y) = x f(y) + g(y)$

Keeping x fixed and integrating with respect to y, we get

$$z + x^3 y^3 - \frac{1}{4}\cos(x - y) = x \int f(y) dy + \int g(y) dy + h(x)$$

Denoting $\int f(y) dy = u(y)$, $\int g(y) dy = v(y)$, we get the required solution,

i.e., $z = \frac{1}{4}\cos(x - y) - x^3 y^3 + x u(y) + v(y) + h(x)$

Example 3: Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$.

Solution: Taking y as constant and integrating the given equation twice partially with respect to x, we get

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{1}{2}\sin(2x + 3y) + f(y)$$

and $\frac{\partial z}{\partial y} = -\frac{1}{4}\cos(2x + 3y) + x f(y) + g(y)$

Again, integrating partially with respect to the above equation, we get

$$z = -\frac{1}{12}\sin(2x + 3y) + x \int f(y) dy + \int g(y) dy + h(x)$$

Taking $\int f(y) dy = u(y)$ and $\int g(y) dy = v(y)$, we get

$$z = \frac{-1}{12}\sin(2x + 3y) + x u(y) + v(y) + h(x)$$

Example 4: Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ when $x = 0, z = e^y$ and $\frac{\partial z}{\partial x} = 1$.

Solution: If z is a function of x alone, then the solution is $z = c_1 \sin x + c_2 \cos x$, where c_1 and c_2 are constants.

Since z is a function of x and y, c_1 and c_2 are functions of y.

Hence, the solution of the given equation is

$$z = f(y) \sin x + g(y) \cos x$$

$$\frac{\partial z}{\partial x} = f(y) \cos x - g(y) \sin x$$

when $x = 0, z = e^y$, i.e., $e^y = g(y)$

and $x = 0, \frac{\partial z}{\partial x} = 1$, i.e., $1 = f(y)$

Hence, the required solution is $z = \sin x + e^y \cos x$.

Example 5: Solve $\frac{\partial^2 z}{\partial y^2} = z$, where, $y = 0, z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$.

Solution: If z is a function of y alone, the solution of given equation is $z = c_1 e^y + c_2 e^{-y}$, where c_1 and c_2 arbitrary constants.

Hence, z is a function of x and y.

$\therefore$ c_1 and c_2 are arbitrary functions of x

Hence, the solution of given equation is $z = f(x)e^y + g(x)e^{-y}$

when $y = 0, z = e^x$, i.e., $e^x = f(x) + g(x)$ ...(1)

Also given, when $y = 0, \frac{\partial z}{\partial y} = e^{-x}$

$=> e^{-x} = f(x) - g(x)$ ...(2)

Solving (1) and (2), we get

$f(x) = \frac{(e^x + e^{-x})}{2}$, and $g(x) = \left(\frac{e^x - e^{-x}}{2}\right)$

We have $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$=> f(x) = \cosh x$ and $g(x) = \sinh x$

$\therefore$ The solution is $z = e^y \cosh x + e^{-y} \sinh x$

Exercise:

Solve the following partial differential equations.

(1) $\frac{\partial^2 z}{\partial x \partial y} = 2x + 2y$

(2) $\frac{\partial^2 z}{\partial x \partial y} = xy^2$

(3) $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$, given when $y = 0, z = x^2$ and when $x = 1, z = \cos y$

Answers:

(1) $z = xy(x + y) + f(y) + \phi(x)$

(2) $z = \frac{x^2 y^3}{6} + f(y) + \phi(x)$

(3) $z = \frac{1}{6}x^2 y^2 + \cos y - \frac{1}{6}y^2 - 1 + x^2$.

Linear Equations of the First Order

A linear partial differential equation of the first order is of the form,

$Pp + Qq = R$ ...(1)

where, P, Q and R are the functions of x, y and z. This equation is known as Lagrange's linear partial differential equation and the solution is $\phi(u, v) = 0$ or $u = f(v)$.

Solution of Lagrange's Linear Partial Differential Equation

To obtain the solution of (1), we have the following rule:

(a) Write equation (1) in the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

(b) Solve these simultaneous equation by the method of grouping, giving $u = a$ and $v = b$ as its solutions.

(c) Write the solution as $\phi(u, v) = 0$, or $u = f(v)$.

Solution of Linear Partial Differential Equation Involving n Variables

Consider the equation

$$p_1 \frac{\partial z}{\partial x_1} + p_2 \frac{\partial z}{\partial x_2} + \ldots \ldots \ldots \ldots \ldots \ldots + p_n \frac{\partial z}{\partial x_n} = R \qquad \ldots(1)$$

First find the equations

$$\frac{dx_1}{p_1} = \frac{dx_2}{p_2} = \ldots \ldots \ldots \ldots \ldots \ldots + \frac{dx_n}{p_n} = \frac{dz}{R} \qquad \ldots(2)$$

and obtain an n independent solution of equation (2).

Let these solutions be,

$$u_1 = c_1, u_2 = c_2 \ldots \ldots \ldots \ldots \ldots u_n = c_n$$

then $\phi(u_1, u_2 \ldots \ldots \ldots \ldots u_n) = 0$ is the solution of equation (1), where ϕ is any arbitrary function.

Equation (2) is called the subsidiary equation.

Example Problems

Example 1: Solve $px + qy = z$.

Solution: The subsidiary equations are $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

Taking $\frac{dx}{x} = \frac{dy}{y}$ and integrating and simplifying, we get

$\log x = \log y + \log c_1$

$\log\left(\frac{x}{y}\right) = \log c_1 \Rightarrow \frac{x}{y} = c_1$

Now, taking $\frac{dy}{y} = \frac{dz}{z}$ and integrating, we get

$\log y = \log z + \log c_2 \Rightarrow \frac{y}{z} = c_2$

Hence, the general solution is

$f(c_1, c_2) = 0$, i.e., $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$, where f is arbitrary.

Example 2: Solve $yzp + zxq = xy$.

Solution: This equation is the form $Pp + Qq = R$. Hence, its auxiliary equation is

$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

Taking $\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow xdx = ydy \Rightarrow \int xdx - \int ydy = c_1$, we get

$x^2 - y^2 = c_1$

Similarly, taking $\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow ydy - zdx = 0$, we get

$$=> y^2 - z^2 = c_2$$

$\therefore$ The general solution of the given partial differential equation is

$F(u, v) = 0 => F(x^2 - y^2, y^2 - z^2) = 0$

where F is an arbitrary function.

Example 3: Solve $(y + z)p + (x + z)q = x + y$.

Solution: The subsidiary equation is $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$

which can be written as:

$$\frac{dx+dy+dz}{2x+2y+2z} = \frac{dx-dy}{-(x-y)} = \frac{dx-dz}{-(x-z)}$$

Taking the first two ratios and integrating, we get

$$-\frac{1}{2}\int \frac{dx+dy+dz}{x+y+z} = -\int \frac{dx-dy}{x-y}$$

$$\log(x + y + z) = -2\log(x - y) + \log c_1$$

$$\log [(x + y + z)(x - y)^2] = \log c_1$$

$$=> (x + y + z)(x - y)^2 = c_1$$

Similarly, taking last two ratios and integrating, we get

$$\log(x - y) - \log(x - z) = \log c_2$$

$$=> \frac{x-y}{x-z} = c_2$$

Hence, the required solution is $F\left[(x + y + z)(x - y)^2, \frac{x-y}{x-z}\right] = 0$

or $x - y = (x - z)f[(x + y + z)(x - y)^2]$

Example 4: Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$.

Solution: The subsidiary equations are:

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

Using multipliers x, y and z, we get

Each fraction = $\frac{xdx+ydy+zdz}{x^2(z^2-y^2)+y^2(x^2-z^2)+z^2(y^2-x^2)} = \frac{xdx+ydy+zdz}{0}$

$$=> xdx + ydy + zdz = 0$$

On integration, we get

$$x^2 + y^2 + z^2 = c_1$$

Again, $\frac{dx/x}{z^2-y^2} = \frac{dy/y}{x^2-z^2} = \frac{dz/z}{y^2-x^2}$

$$= \frac{(dx/x)+(dy/y)+(dz/z)}{z^2-y^2+x^2-z^2+y^2-x^2} = \frac{(dx/x)+(dy/y)+(dz/z)}{0}$$

$$=> \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

On integration, we get

$$\log x + \log y + \log z = \log c_2$$

$$=> \log(xyz) = \log c_2 => xyz = c_2$$

$\therefore$ The given solution is $xyz = f(x^2 + y^2 + z^2)$

Example 5: Solve $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

Solution: The subsidiary equation of given Lagrange's linear equation is

$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx}$

Taking x, y, z as multipliers, we have

Each of the above fraction $= \frac{xdx+ydy+zdz}{0}$

$xdx + ydy + zdz = 0$

Integrating, we get

$x^2 + y^2 + z^2 = c_1$,

Taking two and three relations, we get

$\frac{dy}{y+z} = \frac{dz}{y-z}$

$=> (y - z)dy = (y + z)dz$

$=> ydy - zdz - (zdy + ydz) = 0$

$=> d\left[\frac{y^2}{2} - \frac{z^2}{2} - zy\right] = 0$

Integrating, we get

$y^2 - z^2 - 2yz = c_2$

The solution of given differential equation is $F(x^2 + y^2 + z^2, y^2 - 2yz - z^2) = 0$

Example 6: Solve $(mz - ny)p + (nx - lz) = ly - mx$.

Solution: The auxiliary equation of the given equation is

$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$

Using multipliers x, y and z, we get

Each of the fraction $= \frac{xdx+ydy+zdz}{0}$

which gives $xdx + ydy + zdz = 0$

On integrating, we get

$x^2 + y^2 + z^2 = c_1$

Again, using multiplier l, m and$n$ we get

$ldx + mdy + ndz = 0$

On integrating, we have

$lx + my + nz = c_2$

Hence, the required solution is $F(x^2 + y^2 + z^2, lx + my + nz) = 0$

Example 7: Solve $x(y - z)p + y(z - x)q = z(x - y)$.

Solution: The auxiliary equations are:

$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

Since $x(y-z)+y(z-x)+z(x-y)=0$, we get

$dx+dy+dz=0$

On integrating, we get

$x+y+z=c_1$

We have $(y-z)+(z-x)+(x-y)=0$

The given equation can be written as

$$\frac{dx/x}{y-z}=\frac{dy/y}{x-y}=\frac{dx/x+dy/y+dz/z}{0}$$

$$\therefore\ \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

On integrating, we get

$\log x+\log y+\log z=\log c$

$=>\log xyz=\log c=>xyz=c$

Hence, the general solution is $F(x+y+z,xyz)=0$.

Example 8: Solve $(y+zx)p-(x+yz)q=x^2-y^2$.

Solution: The auxiliary equation of the above equation is

$$\frac{dx}{y+zx}=\frac{dy}{-(x+yz)}=\frac{dz}{x^2-y^2}$$

Choosing multipliers as x,y and $-z$, we have

Each fraction $=\frac{xdx+ydy+zdz}{0}$

$=>xdx+ydy+zdz=0$

On integration, we get

$x^2+y^2+z^2=c_1$

Similarly, choosing multipliers as, y,x and 1, we get

Each fraction $=\frac{ydx+xdy+dz}{0}$

$=>(ydx+xdy)+dz=0=>d(xy)+dz=0$

Integrating, we get

$xy+z=c_2$

Hence, required solution is $F(x^2+y^2+z^2,xy+z)=0$.

Example 9: Solve $p\tan x+q\tan y=\tan z$.

Solution: The subsidiary equations are $\frac{dx}{\tan x}=\frac{dy}{\tan y}=\frac{dz}{\tan z}$

From the first two members, we have,

$\cot x\,dx=\cot y\,dy$

Integrating, we have

$\log\sin x=\log\sin y+\log c_1$

$$\frac{\sin x}{\sin y}=c_1$$

From the last two members, we have,

$\cot y\, dy = \cot z\, dz$

On integration, we get

$\frac{\sin y}{\sin z} = c_2$

Thus, the general solution is $F\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$.

Example 10: Solve $x^2(z-y)p + y^2(x-z)q = z^2(y-x)$.

Solution: The subsidiary equations are:

$$\frac{dx}{x^2(z-y)} = \frac{dy}{y^2(x-z)} = \frac{dz}{z^2(y-x)}$$

or $\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$

Each fraction $= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$

And also $= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$\therefore\ \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$ and $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

On integration, we get

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$ and $\log x + \log y + \log z = \log c_2$

i.e., $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$ and $xyz = c_2$

Hence, required solution is $F\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$.

Exercise:

(1) Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.

(2) Solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x-y)$.

(3) Solve $(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x+y)$.

(4) Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.

(5) Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$.

Answers:

(1) $F(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$

(2) $F\left(x - y - z, {(x^2 - y^2)}/{z^2}\right) = 0$

(3) $F\left(x - y - z, \frac{x^2+y^2}{z^2}\right) = 0$

(4) $F\left(\frac{x^2+y^2+z^2}{z}, \frac{y}{z}\right) = 0$

(5) $F(x + y + z, xyz) = 0$

Non-linear Equations of First Order

Definition: The equations which involve p and q other than in the first degree are called non-linear partial differential equations of first order. For such equations, the complete solution consists of only two arbitrary constants (i.e., equal to the number of independent variables involved) and the particular integral is obtained by assigning particular values to the constants. The first order non-linear partial differential equations are reducible into four standard forms:

Type I: Non-linear equations of the form $f(p,q) = 0$

Solution: Let the required solution be

$$z = ax + by + c \qquad \ldots(1)$$

Then $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$

Substituting these values in $f(p,q) = 0$, we get

$$f(a,b) = 0$$

From this, we can obtain b in terms of a.

Let $b = \phi(a)$. Then the required solution is $z = ax + \phi(a)y + c$.

Definitions

1. **Complete integral:** A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution of the given equation.
2. **Particular integral:** A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a particular integral.
3. **Singular integral:** Let $f(x,\ y,\ z,\ p,\ q) = 0$ be a partial differential equation whose complete integral is

$$\phi(x,y,z,a,b) = 0 \qquad \ldots(1)$$

Differentiating (1) partially with respect to a and b and then equate to zero, we get

$$\frac{\partial \phi}{\partial a} = 0 \qquad \ldots(2)$$

and

$$\frac{\partial \phi}{\partial b} = 0 \qquad \ldots(3)$$

Eliminating a and b by using equations (1), (2) and (3), the eliminant of a and b is called singular integral.

Example Problems

Example 1: Solve $p^2 + q^2 = npq$.

Solution: Let the required solution is $z = ax + by + c$ …(1)

Then $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$ …(2)

Substituting (2) in the given equation, we get

$$a^2 + b^2 = nab \qquad \ldots(3)$$

To find complete integral, we have to eliminate any one of the arbitrary constants from (1).

From (3), we have

$b^2 - nab + a^2 = 0$ which is a quadratic equation in b

$\therefore b = \frac{na \pm \sqrt{n^2a^2 - 4a^2}}{2} = \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right]$ ...(4)

Substituting (4) in (1), we get

$z = ax + \frac{ay}{2}\left[n \pm \sqrt{n^2 - 4}\right] + c$

To find general integral, we take

$c = f(a)$

which gives $z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right]y + f(a)$ ...(5)

Differentiating partially with respect to a, we get

$0 = x + \frac{1}{2}\left(n + \sqrt{n^2 - 4}\right)y + f'(a)$ ...(6)

Eliminating $'a'$ from equations (5) and (6), we get general integral.

Example 2: Solve $p^2 + q^2 = m^2$.

Solution: The given problem is of the form $f(p, q) = 0$

Let the complete solution be $z = ax + by + c$

where $f(a, b) = 0 => a^2 + b^2 = m^2 => b = \sqrt{m^2 - a^2}$

Thus, the complete solution of the given differential equation is

$z = ax + \sqrt{m^2 - a^2}y + c$ ...(1)

which contains two arbitrary constants.

In order to get the general solution, put $c = f(a)$ in (1), so that

$z = ax + \sqrt{m^2 - a^2}y + f(a)$ ...(2)

Differentiating (2) with respect to a, we get

$0 = x - \frac{a}{\sqrt{m^2 - a^2}}y + f'(a) = 0$ ...(3)

Now, eliminating $'a'$ from the above equations (2) and (3), we get general solution.

In particular, when $c = f(a) = 0$, the elimination of $'a'$ gives us $z^2 = m^2(x^2 + y^2)$ which is a particular solution of the given equation.

Example 3: Solve $pq = 1$.

Solution: The complete solution in $z = ax + by + c$ ...(1)

$\therefore p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$

$\therefore ab = 1$ or $b = \frac{1}{a}$ [putting $a = p$ and $b = q$ in $pq = 1$]

Putting $b = \frac{1}{a}$ in (1), we get

$z = ax + \frac{1}{a}y + c$

Example 4: Solve $x^2p^2 + y^2q^2 = z^2$.

Solution: The given equation can be written as $\left(\frac{x}{2}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial q}{\partial y}\right)^2 = 1$ ...(1)

so that it reduces to standard form I.

Now, set $\frac{dx}{x} = dX, \frac{dy}{y} = dY, \frac{dz}{z} = dZ$

so that $X = \log x, Y = \log y, Z = \log z$ and

Then $px = \frac{\partial z}{\partial X}, py = \frac{\partial z}{\partial Y}$

$\therefore \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial X} = \frac{1}{z} \cdot \frac{\partial z}{\partial X} = \frac{1}{z} \cdot px = \frac{x}{z}\frac{\partial z}{\partial x}$

Similarly, $\frac{\partial Z}{\partial Y} = \frac{y}{z} \cdot \frac{\partial z}{\partial y}$

Substituting these values in (1), we get

$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$ ...(2)

which is of the above form, i.e., of the form $f\left(\frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y}\right) = 0$.

Hence, a complete integral or complete solution of (2) is given by

$z = aX + bY + C$, where $a^2 + b^2 = 1$

$\therefore$ Complete solution of (1) is given by

$\log z = a \log x + b \log y + c$

$= a \log x \pm \sqrt{1 - a^2} \log y + c$ which is a required solution

Exercise:

Find the complete solution of:

(1) $pq + p + q = 0$

(2) $z = px + qy + \sqrt{1 + p^2 + q^2}$

(3) $z = px + qy + p^2y^2$

Answers:

(1) $z = ax - \frac{ay}{1+a} + b$

(2) $z = ax + by + \sqrt{1 + a^2 + b^2}$

(3) $z = ax + by + a^2b^2$

Form II: Equations of the form $f(z, p, q) = 0$, i.e., equations not containing x and y

To solve such equation, we have three steps:

Step 1: Set $u = x + ay$ and put $p = {dz}/{du}, q = a\left({dz}/{du}\right)$ in the given equation.

Step 2: Solve the resulting ordinary differential equation in z and u.

Step 3: Replace u by $x + ay$.

Example Problems

Example 1: Solve $z^2 = 1 + p^2 + q^2$.

Solution: The given equation is of the form $f(z,p,q)=0$

Let $z=f(u)$ where $u=x+ay$ be the solution of the given equation.

Then $p=\frac{dz}{du}$ and $q=a\frac{dz}{du}$

Substituting these values of p and q in the given equation, we get

$z^2=1+\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2$ or

$\sqrt{1+a^2}\frac{dz}{du}=\sqrt{z^2-1}$

Integrating $\int\frac{dz}{\sqrt{z^2-1}}=\frac{1}{\sqrt{1+a^2}}\int du+c$ or $\cosh^{-1}z=\frac{u}{\sqrt{1+a^2}}+c$, we get

the solution is $z=\cosh\frac{x+ay+c}{\sqrt{1+a^2}}$

using $u=x+ay$

Example 2: Solve $p(1+q)=qz$.

Solution: Let $u=x+ay$ so that $p=\frac{dz}{du}$ and $q=a\left(\frac{dz}{du}\right)$,

So, the given equation can be written as $\frac{dz}{du}\left(1+a\frac{dz}{du}\right)=az\frac{dz}{du}$

Separating the variables and integrating, we get

$\log(az-1)=u+b$

$=x+ay+b$ which is the required solution.

Example 3: Solve $z=p^2+q^2$.

Solution: Given equation is $z=p^2+q^2$

Let $u=x+ay$ so that $p=\frac{dz}{du}$ and $q=a\frac{dz}{du}$.

So, the given equation can be written as:

$z=\left(\frac{dz}{du}\right)^2+\left(\frac{adz}{du}\right)^2$

$=(1+a^2)\left(\frac{dz}{du}\right)^2$

$=>\frac{dz}{du}=\sqrt{\frac{z}{1+a^2}}$ or $\frac{dz}{\sqrt{z}}=\frac{du}{\sqrt{1+a^2}}$ (variables and separable)

Integrating $\int\frac{dz}{\sqrt{z}}=\frac{1}{\sqrt{1+a^2}}\int du+c$, we get

$2\sqrt{z}+\frac{u}{\sqrt{1+a^2}}+c$

or $2\sqrt{z}=\frac{x+ay}{\sqrt{1+a^2}}+c$ is required solution.

Example 4: Solve $z^2(p^2+q^2+1)=c^2$.

Solution: Put $u=x+ay$ so that $p=\frac{dz}{du}$ and $q=a\frac{dz}{du}$

Substituting these in the given equation, we get

$z^2\left[\left(\frac{dz}{du}\right)^2+\left(\frac{adz}{du}\right)^2+1\right]=c^2$

$\left(\frac{dz}{du}\right)^2[1+a^2]+1=\frac{c^2}{z^2}$

$$\left(\frac{dz}{du}\right)^2 [1 + a^2] = \frac{c^2}{z^2} - 1 = \frac{c^2-z^2}{z^2}$$

$$\left(\frac{dz}{du}\right)^2 = \frac{c^2-z^2}{z^2(1+a^2)}$$

$$\frac{dz}{du} = \sqrt{\frac{c^2-z^2}{z^2(1+a^2)}}$$

$$dz\frac{z}{\sqrt{c^2-z^2}} = \frac{du}{\sqrt{1+a^2}}$$

Integrating, we get

$$\sqrt{1+a^2}\sqrt{c^2-z^2} = u + b$$

Replacing u by $x + ay$, we get the required solution, i.e.,

$$\sqrt{1+a^2}\sqrt{c^2-z^2} = x + ay + b$$

Example 5: Solve $q^2 = z^2p^2(1-p^2)$.

Solution: Given $q^2 = z^2p^2(1-p^2)$ …(1)

Let $u = x + ay$, so that $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$

Substituting these in the given equation, we get

$$\left(\frac{adz}{du}\right)^2 = z^2\left(\frac{dz}{du}\right)^2\left[1-\left(\frac{dz}{du}\right)^2\right]$$

$$\Rightarrow a^2 = z^2\left[1-\left(\frac{dz}{du}\right)^2\right]$$

$$\frac{a^2}{z^2} = 1 - \left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = 1 - \frac{a^2}{z^2} = \frac{z^2-a^2}{z^2}$$

or $\frac{dz}{du} = \frac{\sqrt{z^2-a^2}}{z}$

$\frac{z}{\sqrt{z^2-a^2}}dz = du$ (variables and separable)

Integrating $\int \frac{z}{\sqrt{z^2-a^2}}dz = \int du + c$, we get

$$\int \frac{2zdu}{\sqrt{z^2-a^2}} = 2\int du + c$$

$$\Rightarrow 2\sqrt{z^2-a^2} = 2u + c$$

$$\Rightarrow \sqrt{z^2-a^2} = u + c \Rightarrow z^2 - a^2 = (u+c)^2$$

$$\Rightarrow z^2 = a^2 + (u+c)^2$$

Substituting $u = x + ay$, we get required solution, i.e.,

$$z^2 = a^2 + (x + ay + c)^2$$

Example 6: Solve $p^2z^2 + q^2 = p^2q$.

Solution: Let $u = x + ay$ so that $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$

Substituting these in the given equation, we get

$$\left(\frac{dz}{du}\right)^2 z^2 + \left(\frac{adz}{du}\right)^2 = \left(\frac{dz}{du}\right)^2\left(a\frac{dz}{du}\right)$$

$\left(\frac{dz}{du}\right)^2 [z^2 + a^2] = \left(\frac{dz}{du}\right)^2 \left(a\frac{dz}{du}\right)$

$du = \frac{a}{z^2+a^2} dz$

Integrating $u = a \int \frac{1}{z^2+a^2} dz + c$, we get

$$u = \tan^{-1}\left(\frac{z}{a}\right) + c => u - c = \tan^{-1}\frac{z}{a}$$

$\tan(u - c) = {}^{z}/_{a}$

Substituting $u = x + ay$, we get required solution, i.e.,

$a\tan(x + ay - c) = z$

Exercise:

Solve:

(1) $p^2 + pq = z^2$

(2) $ap + \mathrm{b}q + cz = 0$

(3) $p^3 = q^2$

Answers:

(1) $\log z = \frac{1}{\sqrt{1+a}}(x + ay) + c$

(2) $\log z = \frac{-(cx+cay+ce)}{a+b\alpha}$

(3) $uz = a(x + ay + b)^2$

Standard form III: (Variables separable) $f(x, y) = F(y, q)$, i.e., the equations in which the variable z does not appear and the terms containing p and x can be separated from those containing y and q.

Method of Solution: To obtain a solution of such an equation, we proceed as follows:

Let $f(x, y) = F(y, q) = a$ and solve these for p and q to get

$p = \phi(x, a), q = \psi(y, a)$

Since $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$

We have $\int dz = \int(pdx + qdy)$

$=> z = \int \phi(x, a)dx + \int \psi(y, a)dy + b$

which is the required complete solution containing two constants a and b.

Example Problems

Example 1: Solve $q - p + x - y = 0$.

Solution: The given equation can be written as $q - y = p - x$

Let $p - x = q - y = a$ so that $p = x + a$ and $q = y + a$, and the complete solution is

$$z = \int(x + a)dx + \int(y + a)dy + b$$

or $2z = (x + a)^2 + (y + a)^2 + b$

Example 2: Solve $pe^y = qe^x$.

Solution: The given equation can be written as

$pe^{-x} = qe^{-y}$

Let $pe^{-x} = qe^{-y} = a$, so that $p = ae^x, q = ae^y$, and the complete solution is

$z \int ae^x dx + \int ae^y dy + b$

$=> z = ae^x + ae^y + b$

Example 3: Solve $p^2 + q^2 = x + y$.

Solution: The given equation can be written as

$p^2 - x = y - q^2 = a$

so that $p = \sqrt{a + x}$ and $q = \sqrt{y - a}$

Substituting these values in $dz = pdx + qdy$ and integrating, we get

$z = \frac{2}{3}(a + x)^{3/2} + \frac{2}{3}(y - a)^{3/2} + b$ is the required complete solution.

Example 4: Solve $yp = 2yx + \log q$.

Solution: The given equation can be written as

$p = 2x + \frac{1}{y}\log q$

$=> p - 2x = \frac{1}{y}\log q$

Let $p - 2x = \frac{1}{y}\log q = a$

$=> p = 2x + a, q = e^{ay}$

and the complete solution is

$z = \int(2x + a)dx + \int e^{ay} dy + b$

$= x^2 + ax + \frac{e^{ya}}{a} + b$

$=> az = ax^2 + a^2x + a^{ey} + ab$

Example 5: Solve $z^2(p^2 + q^2) = x^2 + y^2$.

Solution: We write the given equation as:

$\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2$ …(1)

Put $zdz = dZ$, so that,

$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = z\frac{\partial z}{\partial x} = P$

$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = z\frac{\partial z}{\partial y} = Q$

and equation (1) taken of the form:

$P^2 + Q^2 = x^2 + y^2$ or $P^2 - x^2 = y^2 - Q^2 = a$

Thus, $P = \sqrt{x^2 + a}$ and $Q = \sqrt{y^2 - a}$ and

$dz = Pdx + Qdy$ which on integration gives the relation

$Z = \frac{1}{2}x\sqrt{x^2 + a} + \frac{1}{2}a\log\left(x + \sqrt{x^2 + a}\right) + \frac{1}{2}y\sqrt{y^2 - a} - \frac{1}{2}a\log\left(y + \sqrt{y^2 - a}\right) + b$

∴ The complete solution is

$$z^2 = x\sqrt{x^2+a} + y\sqrt{y^2-a} + a\log\frac{x+\sqrt{x^2+a}}{y+\sqrt{y^2-a}} + b$$

Example 6: Solve $z(p^2 - q^2) = x - y$.

Solution: The given equation can be written as:

$$\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2 = x - y$$

Taking $\sqrt{z}dz = dZ$, i.e., $Z = \frac{2}{3}z^{3/2}$, we get

$$\left(\frac{\partial Z}{\partial x}\right)^2 - \left(\frac{\partial Z}{\partial y}\right)^2 = x - y$$

$=> P^2 - Q^2 = x - y$, where $P = \frac{\partial Z}{\partial x}, Q = \frac{\partial Z}{\partial y}$

$=> P^2 - x = Q^2 - y$

Let $P^2 - x = Q^2 - y = a$

such that $P = \sqrt{a+x}, Q = \sqrt{a+y}$,

then the complete solution is:

$$z = \int \sqrt{a+x}dx + \int \sqrt{y+a}dy + b$$

$$z = \frac{(a+x)^{3/2}}{3/2} + \frac{(y+a)^{3/2}}{3/2} + b$$

or $z^{3/2} = (x+a)^{3/2} + (y+a)^{3/2} + c$

Exercise:

Solve:

(1) $p - q = x^2 + y^2$

(2) $\sqrt{p} + \sqrt{q} = x + y$

(3) $p - x^2 = q + y^2$

(4) Solve $q^2 - p = y - x$

Answers:

(1) $z = \frac{1}{3}(x^3 - y^3) + a(x+y) + b$

(2) $3z = (x+a)^3 + (y-a)^3 + b$

(3) $z = \frac{x^3}{3} + ax + ay - \frac{y^3}{3} + c$

(4) $z = \frac{(a+x)^2}{2} + \frac{2}{3}(a+y)^{3/2} + c$

Standard form IV: Non-linear equations of the form $z = px + qy + f(p,q)$ (Clairaut's equation) Equations of the form $z = px + qy + f(p,q)$

An equation analogous to Clairaut's ordinary differential equation $y = px + f(p)$

The complete solution of the equation,

$$z = px + qy + f(p,q) \qquad \ldots(1)$$

is $z = ax + by + f(a,b)$

Let the required solution is $z = ax + by + c$

Then $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$ and putting for a and b in (1).

Example Problems

Example 1: Solve $z = px + qy + pq$.

Solution: Putting $a = p$ and $b = q$ in the given equation, we get the complete solution as:

$$z = ax + by + ab$$

Example 2: Solve $z = px + qy - 2\sqrt{pq}$.

Solution: Putting $a = p$ and $b = q$ in the given equation, we get the complete solution,

i.e., $z = ax + by - 2\sqrt{ab}$

Example 3: Solve $pqz = p^2(qx + p^2) + q^2(py + q^2)$.

Solution: $pqz = p^2q\left(x + \frac{p^2}{q}\right) + q^2p\left(y + \frac{q^2}{p}\right)$

$=> z = P\left(x + \frac{p^2}{q}\right) + q\left(y + \frac{q^2}{p}\right)$

$=> z = px + qy + \frac{p^3}{q} + \frac{q^3}{p}$

Putting $a = p$ and $b = q$ in the above equation, we get the complete solution,

i.e., $z = ax + by + \frac{a^3}{q} + \frac{b^3}{p}$

Exercise:

Solve:

(1) $(p + q)(z - px - qy) = 1$

(2) $z = px + qy + \sqrt{p^2 + q^2 + 1}$

(3) $z = px + qy + p^2q^2$

Answers:

(1) $z = ax + by + \frac{1}{a+b}$

(2) $z = ax + by + \sqrt{a^2 + b^2 + 1}$

(3) $z = ax + by + a^2b^2$

Charpit's Method

If the given equation cannot be reduced to any of the above four types of first order non-linear partial differential equations, then we use a method introduced by Charpit for solving all the partial differential equations of the first order. This method is known as Charpit's method.

Consider the equation

$$F(x, y, z, p, q) = 0 \quad \ldots(1)$$

Since z depend on x and y, we have,

$$dZ = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = Pdx + Qdy \quad \ldots(2)$$

Now, if we can find another relation between x, y, z, p, q such that,

$$f(x, y, z, p, q) = 0 \quad \ldots(3)$$

Then solving (1) and (3) for p and q and substituting in equation (2), this will give the solution provided equation (2) is integrable. To determine f, differentiate equation (1) and (3) with respect to x and y so that,

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\cdot\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\cdot\frac{\partial q}{\partial x}=0 \qquad \ldots(4)$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x}=0 \qquad \ldots(5)$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\cdot\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\cdot\frac{\partial q}{\partial y}=0 \qquad \ldots(6)$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y}=0 \qquad \ldots(7)$$

Eliminating $\frac{\partial p}{\partial x}$ from equation (4) and (5), and $\frac{\partial p}{\partial y}$ from equation (6) and (7), we get

$$\left(\frac{\partial F}{\partial x}\cdot\frac{\partial f}{\partial p}-\frac{\partial f}{\partial x}\cdot\frac{\partial F}{\partial p}\right)+\left(\frac{\partial F}{\partial z}\cdot\frac{\partial f}{\partial p}-\frac{\partial f}{\partial z}\cdot\frac{\partial F}{\partial p}\right)p+\left(\frac{\partial F}{\partial q}\cdot\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}\cdot\frac{\partial F}{\partial p}\right)\frac{\partial q}{\partial x}=0$$

$$\left(\frac{\partial F}{\partial y}\cdot\frac{\partial f}{\partial q}-\frac{\partial f}{\partial y}\cdot\frac{\partial F}{\partial q}\right)+\left(\frac{\partial F}{\partial z}\cdot\frac{\partial f}{\partial q}-\frac{\partial f}{\partial z}\cdot\frac{\partial F}{\partial q}\right)q+\left(\frac{\partial F}{\partial p}\cdot\frac{\partial f}{\partial q}-\frac{\partial f}{\partial p}\cdot\frac{\partial F}{\partial q}\right)\frac{\partial p}{\partial y}=0$$

Adding these two equations and using

$$\frac{\partial q}{\partial x}=\frac{\partial^2 y}{\partial x\partial y}=\frac{\partial p}{\partial y}$$

After rearrangement, we find that,

$$\left(-\frac{\partial F}{\partial p}\right)\frac{\partial f}{\partial x}+\left(-\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial y}+\left(-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial z}+\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}\right)\frac{\partial f}{\partial p}+\left(\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}\right)\frac{\partial f}{\partial q}=0 \qquad \ldots(8)$$

Equation (8) is Lagrange's equation with x, y, z, p, q as independent variable and f as the dependent variable. Its solution depends on the subsidinary equations

$$\frac{dx}{\frac{-\partial F}{\partial p}}=\frac{dy}{\frac{-\partial F}{\partial q}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}=\frac{df}{0}$$

An integral of these equations, involving p or q or both, can be taken as the required relation (3), which along with equation (1) will give the values of p and q to make equation (2) integrable.

Example Problems

Example 1: Solve $(p^2+q^2)y=qz$.

Solution: Let $F(x,y,z,p,q)=(p^2+q^2)y-qz=0 \qquad \ldots(1)$

The subsidiary equations are:

$$\frac{dx}{-2py}=\frac{dy}{z-2qy}=\frac{dz}{-qz}=\frac{dp}{-pq}=\frac{dq}{p^2}$$

The last two fractions yield $pdp+qdq=0$

which on integration gives

$$p^2+q^2=c^2 \qquad \ldots(2)$$

In order to solve equations (1) and (2), put $p^2+q^2=c^2$ in equation (1) so that $q=\frac{c^2y}{z}$

Now, substituting this value of q in equation (2), we get

$p = c\sqrt{\frac{z^2-c^2y^2}{z}}$

Hence, $dz = pdx + qdy = \frac{c}{2}\sqrt{(z^2 - c^2y^2)dx} + \frac{c^2y}{z}dy$

$=> zdz - c^2ydy = c\sqrt{z^2 - c^2y^2}dx$

$=> \frac{(1/_2)d(z^2-c^2y^2)}{\sqrt{z^2-c^2y^2}} = cdx$

Integrating, we get the required solution as $z^2 = (a + cx)^2 + c^2y^2$

Example 2: Solve $2zx - px^2 - 2qxy + pq = 0$.

Solution: Let $F = 2zx - px^2 - 2qxy + pq = 0$

The Charpit's auxillary equations are:

$$\frac{dx}{\frac{-\partial F}{\partial p}} = \frac{dy}{\frac{-\partial F}{\partial q}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{df}{0}$$

Here, $\frac{dp}{2z-2ay} = \frac{dq}{0} = \frac{dz}{px^2-pq+2xyq-pq} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dF}{0}$

$\therefore dq = 0 => q = a$

Substituting $q = a$ in the given equation, we have

$2zx - px^2 - 2axy + pa = 0$

$p(x^2 - a) = 2x(z - ay)$

$=> p = \frac{2x(z-ay)}{x^2-a}$

Substituting these values of p, q in dz, we have

$dz = \frac{2x(z-ay)}{x^2-a}dx + ady$

$\frac{dz-ady}{z-ay} = \frac{2xdx}{x^2-a}$

Integrating, we get

$\log(z - ay) = \log(x^2 - a) + \log b$

$=> z - ay = b(x^2 - a)$

$=> z = ay + b(x^2 - a)$ which is the complete integral of the given equation.

Example 3: Solve $p(q^2 + 1) + (b - z)q = 0$.

Solution: Let $F(x, y, z, p, q) = p(q^2 + 1) + (b - z)q = 0$

Using Charpit's auxiliary equations,

$$\frac{dp}{pq} = \frac{dq}{q^2} = \frac{dz}{3pq^2 + p + (b - z)q} = \frac{dx}{q^2 + 1} = \frac{dy}{-z + b + 2pq}$$

(from the given equation, the third fraction reduces to $^{dz}/_{2pq^2}$).

From the first two fractions, after integration, we get

$q = ap$

where a is an arbitrary constant.

This and the given equation determine the values of p and q as:

$$p = \frac{\sqrt{a(z-b)-1}}{a}, q = \sqrt{a(z-b)-1}$$

Substituting these values in $dz = pdx + qdy$, we have

$$dz = \left(\frac{dx}{a} + dy\right)\sqrt{a(z-b)-1}$$

Separating the variables and integrating, we get the solution as:

$$2\sqrt{a(z-b)-1} = x + ay + b$$

Example 4: Find the complete integral of the equation $2(z + xp + yq) = yp^2$.

Solution: $F = 2(z + xp + yq) - yp^2 = 0$

The Charpit's auxiliary equations are:

$$\frac{dp}{2p+2p} = \frac{dq}{2q-p^2+2q} = \frac{dz}{-p(2x-2yp)-2qy} = \frac{dx}{-2x+2yp} = \frac{dy}{-2y}$$

Taking first and fifth ratios, we have

$$\frac{dp}{p} + \frac{2dy}{y} = 0$$

Integrating, we get

$$py^2 = a$$

Substituting $p = \frac{a}{y^2}$ in the given equation, we get

$$z + \frac{ax}{y^2} + yq = \frac{a^2}{2y^3}$$

$$=> q = \frac{-z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}$$

Substituting p, q in $dz = pdx + qdy$, we have

$$dz = \frac{a}{y^2}dx - \frac{z}{y}dy - \frac{ax}{y^3}dy + \frac{a^2}{2y^4}dy$$

or $ydz + zdy = a\left(\frac{ydx-xdy}{y^2}\right) + \frac{a^2}{2y^3}dy$

Integrating, we get

$$yz = \frac{ax}{y} - \frac{a^2}{4y^3} + b$$

$=> z = \frac{ax}{y^2} - \frac{a^2}{4y^4} + \frac{b}{y}$ is the required solution.

Example 5: Solve $p^2 - y^2q = y^2 - x^2$.

Solution: Let $F(x, y, z, p, q) = p^2 - y^2q - y^2 + x^2 = 0$ …(1)

From Charpit's auxiliary equations be

$$\frac{dp}{2x} = \frac{dq}{-2qy-2y} = \frac{dz}{-p(2p)-q(-q^2)} = \frac{dx}{-2p} = \frac{dy}{y^2} \quad \text{…(2)}$$

Taking $\frac{dp}{2x} = \frac{dx}{-2p}$, we get

$$pdp + xdx = 0 => p^2 + x^2 = a^2 \quad \text{…(3)}$$

Solving (1) and (2) for p and q, we get

$p = (a^2 - x^2)^{1/2}$, and $q = a^2y^{-2} - 1$

$dz = pdx + qdy = (a^2 - x^2)^{1/2}dx + (a^2y^{-2} - 1)dy$

Integrating, we get

$$z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{a^2}{y} - y + b$$

Example 6: Solve $z^2 = pqxy$.

Solution: Let Charpit's auxiliary equations be

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{pqy - 2yz} = \frac{dq}{pqx - 2qz} \quad \ldots(1)$$

These give $\frac{xdp + pdx}{-2pxz} = \frac{ydq + qdy}{-2qyz}$

or $\frac{d(p/x)}{px} = \frac{d(qy)}{qy}$

Integrating, we get

$\log px = \log qy + \log b^2$

$=> px = qy(b^2)$ …(2)

Solving the equations (1) and (2) for p and q, we get

$p = \frac{bz}{x}$ and $q = \frac{z}{by}$

Putting these values in $dz = pdx + qdy$, we get

$dz = \frac{bz}{x}dx + \frac{z}{by}dy$

or $\frac{dz}{z} = b\frac{dx}{x} + \frac{1}{b}\frac{dy}{y}$

Integrating, we get

$$\log z = b \log x + \frac{1}{b}\log y + \log a$$

or $z = ax^by^{1/b}$ is the required solution.

Example 7: Find the complete integral of $q = 3p^2$.

Solution: The given equation is $F(x, y, z, p, q) = 3p^2 - q = 0$ …(1)

By Charpit's auxiliary equations,

$$\frac{dp}{0 + p \cdot 0} = \frac{dq}{0 + q \cdot 0} = \frac{dz}{-bp^2 + q} = \frac{dx}{bp}$$

We have, $dp = 0 => p = a$ …(2)

Substituting this value in (1), we get

$q = 3a^2$ …(3)

Putting these values in $dz = pdx + qdy$, we get

$dz = adx + 3a^2dy$

$=> z = ax + 3a^2y + b$ is required solution.

Example 8: Solve $px + qy = pq$.

Solution: Let $F = px + qy - pq$ …(1)

By Charpit's auxiliary equations,

$$\frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{dz}{-p(x-p)-q(y-p)} = \frac{dp}{p} = \frac{dq}{q}$$

From the last two terms, we get

$p = aq$ ...(2)

Solving the equations (1) and (2), we get

$q = \frac{y+ax}{a}$ and $p = y + ax$

Putting these values in $dz = pdx + qdy$, we get

$dz = (y + ax)dx + \frac{1}{a}(y + ax)dy$

or $adz = (y + ax)(dy + adx)$

On integrating, we get

$az = \frac{1}{2}(y + ax)^2 + b$ which is required solution.

Example 9: Solve $pxy + pq + qy = yz$.

Solution: Let $F(x, y, z, p, q) = pxy + pq + qy - yz = 0$ ...(1)

By Charpit's auxiliary equations,

$$\frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)} = \frac{dz}{-p(xy+q)-p(p+y)} = \frac{dp}{py+p(-y)} = \frac{dq}{px+q(-y)}$$

From 4^{th} fraction, we get

$dp = 0$ or $p = a$ ...(2)

Solving (1) and (2), for p and q, we get

$p = a, \quad q = \frac{y(z-ax)}{a+y}$

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = adx + \frac{y(z-ax)}{a+y}dy$

or $\frac{dz-adx}{z-ax} = \frac{y}{a+y}dy = \left(1 - \frac{a}{a+y}\right)dy$

On integrating, we get

$\log(z - ax) = y - a\log(a + y) + \log b$

or $(z - ax)(y + a)^a = be^y$

Example 10: Solve $z = p^2x + q^2y$.

Solution: Let $F(x, y, z, p, q) = p^2x + q^2y - z = 0$ ...(1)

By Charpit's auxiliary equations,

$$\frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dp}{p^2-p} = \frac{dq}{q^2-q}$$

These give $\frac{p^2dx+2pxdp}{p^2x} = \frac{q^2dy+2qydy}{q^2y}$

On integrating, we get

$\log p^2x = \log q^2y + \log a$

$\therefore\ p^2x = aq^2y$ ...(2)

Solving equations (1) and (2) for p and q, we get

$$q = \sqrt{\frac{z}{(1+a)y}},\ p = \sqrt{\frac{az}{(1+a)x}}$$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \left\{\frac{az}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy$$

or $\sqrt{1+a}\frac{dz}{\sqrt{z}} = \sqrt{a}\frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}}$

Integrating, we get

$\sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b$ is the required complete solution.

Exercise:

(1) Find the complete integral of the equation $q = z + (px)^2$.
(2) Solve $(p^2 + q^2)y = qz$.
(3) Solve $z = px + qy + p^2 + q^2$.

Answers:

(1) $xz = 2\sqrt{a}\sqrt{x} + ay + b$
(2) $z^2 = a^2y^2 + (ax + b)^2$
(3) $z = ax + by + a^2 + b^2$

Homogeneous Linear Equations with Constant Coefficients

An equation of the form,

$$\frac{\partial^2 z}{\partial x^n} + k_1 \frac{\partial^2 z}{\partial x^{n-1}\partial y} + \ \dots\dots\dots\dots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad \dots(1)$$

where, $k_1, k_2, \dots\dots\dots\dots k_n$ are constants is called a linear homogeneous partial differential equation of order n with constant coefficients.

Equation (1) can be written as:

$$(D^n + k_1 D^n D' + \ \dots\dots\dots\dots + k_n D^n)z = F(x, y) \qquad \dots(2)$$

or $f(D, D')z = F(x, y)$

where, $D^n = \frac{\partial^n}{\partial x^n}, D^n = \frac{\partial^n}{\partial y^n}$ and

$F(D, D') = D^n + a_1 D^{n-1} D' + \ \dots\dots\dots\dots + a_n D^n$

Solution of the Linear Partial Differential Equation with Constant Coefficients

Equation (1) consists of two parts:

(i) Complementary function (C.F.) and
(ii) Particular integral (P.I.)

(i) Method of finding C.F. for $\boldsymbol{F(D, D')Z = F(x, y)}$

Case I: If $m_1, m_2, \dots\dots\dots\dots m_n$ are n distinct roots of the auxiliary equation of $F(D, D')Z = 0$, then its complementary function (C.F.) is

$$z = f_1(y + m_1 x) + f_2(y + m_n x) + \ \dots\dots\dots\dots + f_n(y + m_n x)$$

Case II: When the auxiliary equation has a root m which is repeated twice $'r'$ times, then its C.F. is

$$z = f_1(y+mx) + f_2(y+mx) + \dots\dots\dots + x^{n-1} f_r(y+mx)$$

Note: The auxiliary equation of $F(D, D')Z = 0$ can be obtained by taking $\frac{D}{D'} = m$ or $D = m$, $D' = 1$ in $F(D, D') = 0$.

i.e., A.E. is $f(m, 1) = m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots\dots\dots + a_n = 0$.

(ii) Finding particular integral for $F(D, D')Z = F(x, y)$

Case I: If $F(x, y)$ in the given equation (1), is any function of x, y then

$$\text{P.I.} = \frac{1}{f(D,D')} F(x, y)$$

and to find P.I., resolve $\frac{1}{f(D,D')}$ into partial fractions considering $f(D, D')$ as a function D only and for each partial fraction use the following formula:

$$\frac{1}{D - mp'} F(x, y) = \int F(x, c - mx) dx$$

where c is an arbitrary constant which can be replaced by $y + mx$ after integrable.

Case II: When $F(x, y) = e^{ax+by}$

$$\text{P.I.} = \frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by} \qquad \text{provided } f(a, b) \neq 0$$

Note: If $f(a, b) = 0$ then

$$\frac{1}{f(D,D')} e^{ax+by} = \frac{1}{g(D,D')(bD - aD')^r} e^{ax+by}$$

$$= \frac{1}{g(a,b)} \frac{x^r}{r! b^n} e^{ax+by}$$

Case III: When $F(x, y) = \sin(ax + by)$ or $\cos(ax + by)$

$$\text{P.I.} = \frac{1}{f(D^2, DD', D^2)} \sin(ax + by) = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by)$$

and

$$\text{P.I.} = \frac{1}{f(D^2, DD', D^2)} \cos(ax + by) = \frac{1}{f(-a^2, -ab, -b^2)} \cos(ax + by)$$

Case IV: When $F(x, y) = x^m y^n$, where m and n are positive integers as:

$$\text{P.I.} = \frac{1}{f(D,D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$$

Now, expand $[f(D, D')]^{-1}$ as an infinite series in ascending powers of D or D' using the binomial theorem and then operate on $x^m y^n$ term by term.

Case V: When $F(x, y)$ is a function of $ax + by$, that is $F(x, y) = \psi(ax + by)$,

$\frac{1}{f(D,D')} \psi^{(n)}(ax + by) = \frac{1}{F(a,b)} \psi(ax + by)$, provided $F(a, b) \neq 0$

where $\psi^{(n)}$ is the n^{th} derivative of ψ.

Example Problems

Example 1: Solve $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$.

Solution: The given equation can be written as

$(2D^2 + 5DD' + 2D^2)z = 0$

The auxiliary equation $2m^2 + 5m + 2 = 0$ has the roots $m = -2, \frac{-1}{2}$. Hence, the complete solution is $z = f_1(y - 2x) + f_2\left(y - \frac{1}{2}x\right)$.

Example 2: Solve $\frac{\partial^2 z}{\partial x^2} + 6\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$.

Solution: The given equation can be written as $m^2 + 6m + 9 = 0$ has the roots $m = -3, -3$ and complete solution is $z = f_1(y - 3x) + xf_2(y - 3x)$.

Example 3: Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$.

Solution: The roots of the auxiliary equation of the given equation is $m = 0,\ 1$

$\therefore$ C.F. $= f_1(y) + f_2(y + x)$

and P.I. $= \frac{1}{D^2 - DD'} \cos x \cos 2y$

$= \frac{1}{2}\frac{1}{D^2 - DD'}[\cos(x + 2y) + \cos(x - 2y)]$

Putting $D^2 = 1, DD' = -2$ and $D^2 = -1, DD' = 2$, we get

P.I. $= \frac{1}{2}\cos(x + 2y) - \frac{1}{6}\cos(x - 2y)$

Thus, the complete solution is

$z = f_1(y) + f_2(y + x) + \frac{1}{2}\cos(x + 2y) - \frac{1}{6}\cos(x - 2y)$

Example 4: Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial x \partial y^2} = 0$.

Solution: The auxiliary equation of the given equation $f(D, D')z = (D^3 - 3D^2D' + 2D^3)z = 0$ is $f(m, 1) = m^3 - 3m^2 + 2m = 0$ whose roots are $m = 0, 1, 2$.

Hence, the solution for three distinct roots of auxiliary equation

$z = f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x)$ becomes

$z = f_1(y) + f_2(y + x) + f_3(y + 2x)$

Example 5: Solve $\frac{\partial^4 y}{\partial x^4} - \frac{\partial^4 y}{\partial y^4} = 0$.

Solution: Given equation is $f(D, D')z = (D^4 - D'^4)z = 0$

Auxiliary equation is $f(m, 1) = 0$

$=> m^4 - 1 = 0$

$=> m = -1, 1 \pm i$

Hence, the solution is $z = f_1(y - x) + f_2(y + x) + f_3(y + ix) + f_4(y - ix)$

Example 6: Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = e^{x+2y} + x^3$.

Solution: The given equation can be written in the form $f(D, D')z = e^{x+2y} + x^3$

where $f(D, D') = D^2 - 2DD' + D'^2$

The auxiliary equation of $f(D, D')z = 0$ is $f(m, 1) = m^2 - 2m + 1 = 0 => m = 1, 1$

C.F. $= f_1(y + x) + xf_2(y + x)$

P.I. of $f(D,D')z = e^{x+2y} + x^3$ is

$$= \frac{1}{D^2-2DD'+D'^2} e^{x+2y} + \frac{1}{D^2\left(1-\frac{D'}{D}\right)^2} x^3$$

$$= \frac{1}{1^2-2.1.2+2^2} e^{x+2y} + \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2} x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left[1 + 2\frac{D'}{D} + 3\frac{D'^2}{D^2} + \ldots\ldots\ldots\ldots\right] x^3$$

$$= e^{x+2y} + \frac{1}{D^2}(x^3 + 0) = e^{x+2y} + \frac{x^5}{20}$$

Now, complete integral of the given equation is

$$z = f_1(y+x) + xf_2(y+x) + e^{x+2y} + \frac{x^5}{20}$$

Example 7: Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x+y)$.

Solution: The given equation is $f(D,D')z = \cos(2x+y)$

where $f(D,D') = D^2 - 2DD' + D'^2$

The auxiliary equation of $f(D,D')z = 0$ is $f(m,1) = m^2 + m - 6 = 0 => m = -3, 2$

C.F. $= f_1(y-3x) + xf_2(y+2x)$

P.I. $= \frac{1}{D^2+DD'-6D'^2}\cos(2x+y)$, where $f(a,\ b) = 0$

$$= \frac{1}{(D-2D')(D+3D')}\cos(2x+y)$$

$$= \frac{1}{D-2D'} \cdot \frac{1}{2+3}\sin(2x+y)$$

$$= \frac{1}{5} \cdot \frac{x}{1!}\sin(2x+y)$$

$$= \frac{x}{5}\sin(2x+y)$$

The general solution of the given equation is

$$z = f_1(y-3x) + xf_2(y+2x) + \frac{x}{5}\sin(2x+y)$$

Example 8: Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$.

Solution: Auxiliary equation $f(m,1) = m^2 + m - 6 = 0$

$$=> (m-3)(m+2) => m = -2, 3$$

C.F. $= f_1(y-2x) + f_2(y+3x)$

P.I. $= \frac{1}{(D-2D')(D+3D')} y\cos x$

$$= \frac{1}{D-2D'}\int (c+3x)\cos x\, dx$$

$$= \frac{1}{D-2D'}\left[(c+3x)\sin x - 3\int \sin x\, dx\right]$$

$$= \frac{1}{D-2D'}[y\sin x + 3\cos x] \qquad \because\ c+3x = y$$

$$= \int [(c-2x)\sin x + 3\cos x]dx$$

$$= -(c-2x)\cos x + \int(-2)\cos x\, dx + 3\sin x$$

$$= -y\cos x + \sin x$$

The complete integral is

$$z = f_1(y-2x) + f_2(y+3x) - y\cos x + \sin x$$

Non-homogeneous Linear Partial Differential Equations

If in $\left(D^n + K_1 D^n D' + \cdots \ldots \ldots \ldots + k_n D'^n\right)z = F(x, y)$ or $f(D, D')z = F(x, y)$, the polynomial $f(D, D')$ is not homogeneous, then the above equation is called a non-homogeneous linear partial differential equation. Its complete solution consists of a complementary function and a particular integral. The methods for finding the particular integral are same as discussed above method, and to obtain complementary function, we factorize $f(D, D')$ into factors of the form $D - mD' - c$. Now, to find the solution of $(D - mD' - c)z = 0$, which can be written as:

$$p - mq = cz \qquad \ldots(1)$$

The subsidiary equation is $\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz}$

Its integrals are $y + mx = a, z = be^{ax}$

Taking $b = \phi(a)$, we get the solution of equation (1) as $z = e^{ax}\phi(y + mx)$. The solutions corresponding to the various factors, when added up, give the complementary function of the non-homogeneous linear partial differential equation.

Example Problems

Example 1: Solve $\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right)z = \sin(x + 2y)$.

Solution: $f(D, D') = (D + D')(D + D' - 2)$.

The solution corresponding to the factor $D - mD' - c$ is

$z = e^{ax}\phi(y + mx)$

$\therefore$ C.F. $= \phi_1(y - x) + e^{2x}\phi(y - x)$

and P.I. $= \frac{1}{D^2+2DD'+D'^2-2D-2D'}\sin(x + 2y)$

$= \frac{1}{-1+2(-2)+(-4)-2D-2D'}\sin(x + 2y)$

$= \frac{1}{2(D+D')+9}\sin(x + 2y)$

$= \frac{2(D+D')-9}{4(D^2+2DD'+D^2)-81}\sin(x + 2y)$

$= \frac{2(D+D')-9}{4[-1+2(-2)-4]-81}\sin(x + 2y)$

$= \frac{1}{117}\{2[\cos(x + 2y) + 2\cos(x + 2y)] - 9\sin(x + 2y)\}$

$= \frac{1}{39}[2\cos(x + 2y) - 3\sin(x + 2y)]$

Thus, the complete solution is

$z = \phi_1(y - x) + e^{2x}\phi_2(y - x) + \frac{1}{39}[2\cos(x + 2y) - 3\sin(x + 2y)]$

Rules to Find Complementary Function in Case of Non-homogeneous Linear Partial Differential Equation

Rule I: If $D - m_1D' - c_1, D - m_2D' - c_2, \ldots \ldots \ldots \ldots, D - m_nD' - c_n$ are n distinct linear factors of $f(D, D')$, then its corresponding C.F. of $f(D, D')z = F(x, y)$ is given by:

C.F.$= e^{c_1x}f_1(y + m_1x) + e^{c_2x}f_2(y + m_2x) + \ldots \ldots \ldots \ldots + e^{c_nx}f_n(y + m_nx)$

Rule II: If $D - mD' - c$ is a linear factor of $f(D, D')$, which is repeated k times, then its corresponding C.F. of $f(D, D')z = F(x, y)$ is

C.F.$= e^{cx}f_1(y + mx) + xe^{cx}f_2(y + mx) + \ldots \ldots \ldots \ldots + x^{k-1}e^{cx}f_k(y + mx)$

Rule III: If $f(D, D')$ cannot be resolved into linear factors in D and D', then the equation cannot be integrated by above methods. Hence, in a trial method, we take the solution corresponding to non-resolved factors as $z = \sum Ae^{hx+ky}$ and substitute it in corresponding factors equation to get the values of h, k.

Note: For $f(D, D')z = e^{ax+by}V$, where V is a function of x and y then

$$\text{P.I.} = \frac{1}{f(D,D')} e^{ax+by} V$$
$$= e^{ax+by} \cdot \frac{1}{f(D + a_1 D' + b)} \cdot V$$

Example 2: Solve $(D^2D' - 2DD'^2 + 3DD')z = 0$.

Solution: Given equation can be written as $DD'(D - 2D' - 3)z = 0$

The factors of $f(D, D')$ are $(D - 0D' - 0), (D' - 0.D - 0), (D - 2D' - 3)$

$\therefore$ The complete solution is

$z = e^{0x}f_1(y) + e^{0y}f_2(x) + e^{3x}f_3(y + 2x)$

$\Rightarrow z = f_1(y) + f_2(x) + e^{3x}f_3(y + 2x)$

Example 3: Solve $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$.

Solution: The C.F. corresponding to the linear factor $D - 2D' - 1$ is $z = e^x f_1(y + 2x)$

Since $(D - 2D'^2 - 1)$ cannot be resolved into product of linear factors, we take its corresponding C.F. as $z = \sum Ae^{hx+ky}$ and substituting in $(D - 2D'^2 - 1)z = 0$, we get $(h - 2K^2 - 1)\sum Ae^{hx+ky} = 0$ or $h = 2k^2 + 1$

Hence, its corresponding C.F. is $z = \sum Ae^{(2k^2+1)x+ky}$

$\therefore$ The complete solution of the given equation is

$z = e^x f_1(y + 2x) + \sum Ae^{ky+(2k^2+1)x}$

Example 4: Solve $\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} - z = xy$.

Solution: The given equation can be written symbolically,

$(DD' + D - D' - 1)z = xy$

or $(D - 1)(D' + 1)z = xy$

C.F. $= e^x f_1(y) + e^{-y} f_2(x)$

P.I. $= \frac{1}{(D-1)(D'+1)} xy = -(D - 1)^{-1}(D' + 1)^{-1}xy$

$= -(1 + D + D^2 + \ldots\ldots\ldots\ldots)(1 - D' + D'^2 + \ldots\ldots\ldots\ldots)xy$

$= -(1 + D + D^2 + \ldots\ldots\ldots\ldots)(xy - x)$

$= -xy + x - y + 1$

Now, the complete solution is given by:

$z = e^x f_1(y) + e^{-y} f_2(x) - xy + x - y + 1$

Example 5: Solve $\frac{2\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} - \frac{3\partial z}{\partial y} = 3\cos(3x - 2y)$.

Solution: Here, $f(D, D') = 2DD' + D'^2 - 3D' = (2D + D' - 3)D'$

Now, $(2D + D' - 3)z = 0$ gives us $2p + q = 3z$

The subsidiary equations are:

$\frac{dx}{2} = \frac{dy}{1} = \frac{dz}{3z}$

Solving I and II fractions, we have

$2y - x = a$

Solving II and III fractions, we have

$\log z = 3y + \log b => z = be^{3y}$

Taking $b = \phi(a)$ we have

$z = e^{3y}\phi(2y - x)$

Now, taking the other factor D', we have,

$D'z = 0 => \frac{\partial z}{\partial y} = 0 => z = f_1(x)$

C.F. $= f_1(x) + e^{3y} f_2(2y - x)$

P.I. $= \frac{1}{2DD' + D'^2 - 3D'} 3\cos(3x - 2y)$

$= \frac{1}{-2(3)(-2) - 4 - 3D'} 3\cos(3x - 2y)$

$= \frac{3(8 + 3D')}{64 - 9D'^2} \cos(3x - 2y)$

$= 3\left[\frac{8\cos(3x - 2y) + 6\sin(3x - 2y)}{64 - 9[-4]}\right]$

$= \frac{6}{25}\cos(3x - 2y) + \frac{9}{50}\sin(3x - 2y)$

$\therefore$ The general solution of the given equation is

$z = f_1(x) + e^{3y} f_2(2y - x) + \frac{6}{25}\cos(3x - 2y) + \frac{9}{50}\sin(3x - 2y)$

Example 6: Solve $(D - 3D' - 2)^2 z = 2e^{2x}\tan(y + 3x)$.

Solution: The given equation is $(D - 3D' - 2)^2 z = 2e^{2x}\tan(y + 3x)$

Here, $D - 3D' - 2$ is twice repeated factor of $f(D, D')$

Hence, the C.F. $= e^{2x} f_1(y + 3x) + xe^{2x} f_2(y + 3x)$

P.I. $= \frac{1}{(D - 3D' - 2)^2} 2e^{2x}\tan(y + 3x)$

$= 2e^{2x} \cdot \frac{1}{(D + 2 - 3D' - 2)^2}\tan(y + 3x)$ $\quad \left[\because \frac{1}{f(D, D')} e^{ax + by} \cdot V = e^{ax + by} \cdot \frac{1}{f(D + a, D' + b)} V\right]$

$= 2e^{2x} \cdot \frac{1}{(D - 3D')^2}\tan(y + 3x)$

Here, $f(a, b) = [3 - 3(1)^2] = 0$

Hence using the formula,

$\frac{1}{(bD - aD')^n}\psi(ax + by) = \frac{x^n}{b^n n!}\psi(ax + by)$

We get

P.I. $= 2e^{2x} \cdot \frac{x^2}{2!}\tan(y + 3x)$

$= x^2 e^{2x} \tan(y+3x)$

Now, the complete solution of the given equation is

$z = e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} \tan(y+3x)$

Example 7: Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$.

Solution: The given equation is $\left(D^2 - 2DD' + D'^2\right)z = \sin x$ ...(1)

A.E. is $m^2 - 2m + 1 = 0 => (m-1)^2 = 0 => m = 1, 1$

$\therefore$ C.F. $= f_1(y+x) + x f_2(y+x)$

P.I. $= \frac{1}{D^2 - 2DD' + D'^2} \sin x$

$= \frac{1}{-1 - 2(0) + (0)^2} \sin x$

$= -\sin x$

$\therefore$ The general solution is

$z = f_1(y+x) + x f_2(y+x) - \sin x$

Exercise:

(1) Solve $\left(D^2 - 2DD' + D'^2\right) = (y-1)e^x$.

(2) Solve $\left(D^2 + D'^2\right)z = x^2 y^2$.

(3) Solve $\left(2D^2 - 5DD' + 2D'^2\right)z = 5\sin(2x+y)$.

Answers:

(1) $z = f_1(y-x) + f_2(y+2x) + ye^x$

(2) $z = f_1(y+ix) + f_1(y-ix) + i[f_2(y+ix) - f_2(y+ix)] + \frac{1}{180}(15x^4y^2 - x^6)$

(3) $z = f_1(y+2x) + f_2(2y+x) - \frac{10}{3} x\cos(y+2x)$

Separation of Variables

Consider a partial differential equation of the form

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} = F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \quad \text{...(1)}$$

where, A, B and C are continuous functions of x and y, the derivatives are also continuous and F denotes a polynomial function of $x, y, z, \frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Equation (1) is called hyperbolic if $B^2 - 4AC > 0$.

Parabolic if $B^2 - 4AC = 0$, elliptic if $B^2 - 4AC < 0$.

However, whether an equation is parabolic, ecliptic or hyperbolic, depends upon the values of A, B and C in (1).

Thus, the equation $\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0$ is elliptic, when $x > 0$ and is hyperbolic when $x < 0$.

Partial differential equations of the second order are of great importance in physical problems. The differential equation which satisfies certain conditions at the boundary points is known as boundary value problem. The method used in finding the solutions of boundary value problems involving partial differential equation is known as "separation of variables".

Method of Separation of Variables

Separation of variables is a powerful technique to solve partial differential equation. In this method, we assume the required solution of the given partial differential equation as:

$$z = X(x) \cdot Y(y) \quad \ldots(1)$$

where, $X(x)$ is a function of x alone and $Y(y)$ is a function of y alone. Now, substituting z and its partial derivatives in the given partial differential equation, it reduces to the form:

$$f(X, X', X'' \ldots\ldots\ldots\ldots) = g(Y, Y', Y'' \ldots\ldots\ldots\ldots) \quad \ldots(2)$$

which is separable in X and Y.

∵ The L.H.S. of the equation (2) is a function of X alone and R.H.S. of the equation (2) is a function of Y alone, then each side of (2) must be a constant say k. The equation (2) reduces to the ordinary differential equation,

$$f(X, X', X'' \ldots\ldots\ldots\ldots) = k \quad \ldots(3)$$

and $g(Y, Y', Y'' \ldots\ldots\ldots\ldots) = k$ ...(4)

Now, the complete solution of the given partial differential equation is the product of the solutions of the equations (3) and (4).

Example Problems

Example 1: Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

Solution: We assume the solution of the given equation in the form $z = X(x).Y(y)$

where, X is a function of x alone and Y is a function of y alone.

Now from Z, we get

$$\frac{\partial z}{\partial x} = X'Y, \frac{\partial^2 z}{\partial x^2} = X''y, \frac{\partial z}{\partial y} = XY'$$

Substituting these values in the given differential equation, we get

$$X''Y - 2X'Y + XY' = 0$$

where, $X' = \frac{dX}{dx}$ and $Y' = \frac{dY}{dy}$ and $X'' = \frac{d^2X}{dx^2}$

Separating the variables in the above equation, we get

$$\frac{X''-2X'}{X} = \frac{-Y'}{Y} = k(\text{say})$$

or $X'' - 2X' - kX = 0$ and $Y' + kY = 0$

Auxiliary equation of the above equations are $m^2 - 2m - k = 0$ and $m' + k = 0$

$$m = \frac{2 \pm \sqrt{4+4k}}{2} = 1 \pm \sqrt{1+k} \text{ and } m' = -k$$

The solutions of the above equations are:

$$X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x} \text{ and } Y = c_3 e^{-k_y}$$

Now, the solution of the given differential equation is:

$$z = X(x)\cdot Y(y) = \left[c_1e^{(1+\sqrt{1+k})x} + c_2e^{(1-\sqrt{1+k})x}\right]c_3e^{-ky}$$

$$= \left[b_1e^{(1+\sqrt{1+k})x} + b_2e^{(1-\sqrt{1+k})x}\right]e^{-ky}$$

Example 2: Solve $3\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = 0$ with $z(x,0) = 4e^{-x}$.

Solution: Assume the solution of the given equation as

$$z = X(x)\cdot Y(y) => \frac{\partial z}{\partial x} = X'Y, \frac{\partial z}{\partial y}XY'$$

Substituting $z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ in the given equation, we get

$$3X'Y + 2XY' = 0$$

or $\frac{X'}{X} = \frac{-2Y'}{3Y} = k(\text{say})$

Solving $X' - kX = 0$ and $2Y' + 3kY = 0$, we get

$X(x) = c_1e^{kx}$ and $Y = c_2e^{\frac{-3}{2}ky}$

Now, the solution of the given equation is:

$$z = X(x)\cdot Y(y) = c_1e^{kx}\cdot c_2e^{-3/2ky} = ce^{\frac{-k}{2}(2x-3y)}$$

Given that $z(x,0) = 4e^{-x}$

$=> ce^{kx} = 4e^{-x}$

$=> c = 4, k = -1$

Now, the required particular solution is:

$$z = 4e^{-1/2(2x-3y)}$$

Example 3: Solve $\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial y^2} = 0$.

Solution: We assume the solution of the form

$$z = X(x)\cdot Y(y) \qquad \ldots(1)$$

where, X is a function of x alone and Y is a function of y alone.

From equation (1), the given equation reduced to:

$$X''Y' + 4Y''X = 0 \qquad \ldots(2)$$

where, $X'' = \frac{dX}{dx},\ Y'' = \frac{dY}{dy}$ etc.

Equation (2) can be written as:

$$\frac{X''}{X} = \frac{-4Y''}{Y}$$

Here, X''/X, is a function of x, does not change when y alone changes and $-4\,Y''/Y$ does not change when x alone changes. Hence, the two ratios will be equal only when both are equalto the same constant k, i.e., when,

$$X'' = kX, -4Y'' = kY \qquad \ldots(3)$$

The solutions of these ordinary differential equations (3) for $k > 0$ are:

$$X = a_1e^{\sqrt{k}x} + a_2e^{-\sqrt{k}x}$$

$$Y = a_3 \sin\left(\frac{1}{2}\sqrt{k}y\right) + a_4 \cos\left(\frac{1}{2}\sqrt{k}y\right) \quad \ldots(4)$$

Now, equations (1) and (4) satisfy the given equation for any value of k. Hence, in accordance with the principle of adding of solutions of linear equations, the solution of the given equation can be written as:

$$z = \sum_k X_k Y_k = \sum_k \left\{e^{\sqrt{k}x}\left[a_k \sin\left(\frac{1}{2}\sqrt{k}y\right) + b_k \cos\left(\frac{1}{2}\sqrt{k}y\right)\right] + e^{-\sqrt{k}x}\left[c_4 \sin\left(\frac{1}{2}\sqrt{k}y\right) + d_4 \cos\left(\frac{1}{2}\sqrt{k}y\right)\right]\right\} \quad \ldots(5)$$

where, k may take the values of any finite set of positive numbers. If k takes the values $K_1, K_2 \ldots\ldots\ldots\ldots$, then equation (5), involving an infinite series, is a valid solution within its region of absolute convergence.

If k is negative, the sin-cosine part of the solution would go with X and the exponantal part with Y, when $k = 0$, we have,

$$X = c_1 x + c_2, Y = c_3 y + c_4$$

Exercise:

(1) Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 0$.

(2) Solve $2x\frac{\partial z}{\partial x} = 3y\frac{\partial z}{\partial y}$.

Answers:

(1) $z = \left(Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}\right)e^{2kx}$

(2) $z = Ax^{\frac{k}{2}} \cdot y^{\frac{k}{3}}$

Practical Question Bank with Answers

(1) Form the partial differential equations, by eliminating the arbitrary constants from $z = (x^2 + a)(y^2 + b)$.

Solution: Given equation is $Z = (x^2 + a)(y^2 + b)$ ...(1)

Differentiating equation (1) with respect to x, we get

$$\frac{\partial z}{\partial x} = 2x(y^2 + b)$$

$$p = 2x(y^2 + b)$$

$$\frac{p}{2x} = (y^2 + b) \quad \ldots(2)$$

Differentiating equation (1) with respect to y, we get

$$\frac{\partial z}{\partial y} = 2y(x^2 + a)$$

$$q = 2y(x^2 + a)$$

$$\frac{q}{2y} = (x^2 + a) \quad \ldots(3)$$

From (2) and (3), we get

$$z = \frac{q}{2x} \cdot \frac{p}{2y}$$

$4xyz = pq$

which is the required differential equation.

(2) Find the differential equations of all planes which are at a constant distance r from the origin.

Solution: Equation of all the planar which are at a constant distance r from origin is

$ax + by + cz = r$ ...(1)

with $a^2 + b^2 + c^2 = 1$ ...(2)

From (2), we get

$c = \sqrt{1 - a^2 - b^2}$

Differentiating (1) partially with respect to x, we get

$a + c\frac{\partial z}{\partial x} = 0 \quad \left(\because \frac{\partial z}{\partial x} = p = \frac{-a}{c}\right),\ a + cp = 0$

$a = -pc$ ...(3)

Differentiating (1) partially with respect to y, we get

$b + c\frac{\partial z}{\partial y} = 0\ \left(\frac{\partial z}{\partial y} = q = \frac{-b}{c}\right), b + cq = 0$

$b = -qc$ ...(4)

Substituting a and b in (2), we get

$p^2c^2 + q^2c^2 + c^2 = 1$

$\frac{1}{c} = \sqrt{p^2 + q^2 + 1}$ ...(5)

From (1), we have

$z = \frac{r}{c} - \frac{a}{c}x - \frac{b}{c}y$

From (3), (4) and (5), we have

$z = px + qy + r\sqrt{p^2 + q^2 + 1}.$

(3) Form the partial differential equation by eliminating the arbitrary functions from $F(x^2 + y^2, z - xy) = 0$.

Solution: Given $\phi(x^2 + y^2, z - xy) = 0$

Let $u = x^2 + y^2, v = z - xy$ so that the given relation is:

$F(u, v) = 0$ ...(1)

Differentiating (1) with respect to x and y, we get

$\frac{\partial \phi}{\partial u}(2x) + \frac{\partial \phi}{\partial v}(p - y) = 0$ ...(2)

$\frac{\partial \phi}{\partial u}(2y) + \frac{\partial \phi}{\partial v}(q - x) = 0$ ...(3)

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (2) and (3), we get

$\begin{vmatrix} 2x & p - y \\ 2y & q - x \end{vmatrix} = 0$

$2x(q - x) - 2y(p - y) = 0$

$qx - x^2 - py + y^2 = 0$

$-py + qx = x^2 - y^2$

$py - qx = y^2 - x^2$

which is required partial differential equation.

(4) Solve $x^2(z-y)p + y^2(x-z)q = z^2(y-x)$.

Solution: Given equation can be written as $x^2(z-y)p + y^2(x-z)q = z^2(y-x)$

$$\frac{dx}{x^2(z-y)} = \frac{dy}{y^2(x-z)} = \frac{dz}{z^2(y-x)}$$

or $\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$

Each fraction $= \frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{0}$

and also $= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$

$\therefore\ \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$ and $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

On integration, we get

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$ and $\log x + \log y + \log z = \log c_2$

i.e., $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$ and $xyz = c_2$

Hence, the required solution is $F\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$.

(5) Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$.

Solution: The subsidiary equations are:

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

Using multipliers x, y, z, we get

Each fraction $= \frac{xdx+ydy+zdz}{x^2(z^2-y^2)+y^2(x^2-z^2)+z^2(y^2-x^2)} = \frac{xdx+ydy+zdz}{0}$

$=> xdx + ydy + zdz = 0$

On integration, we get

$x^2 + y^2 + z^2 = c_1$

Again, $\frac{dx/x}{z^2-y^2} = \frac{dy/y}{x^2-z^2} = \frac{dz/z}{y^2-x^2}$

$$= \frac{(dx/x)+(dy/y)+(dz/z)}{z^2-y^2+x^2-z^2+y^2-x^2} = \frac{(dx/x)+(dy/y)+(dz/z)}{0}$$

$=> \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

On integration, we get

$\log x + \log y + \log z = \log c_2$

$=> \log(xyz) = \log c_2 => xyz = c_2$

$\therefore$ The given solution is $xyz = f(x^2 + y^2 + z^2)$

(6) Solve $z(p^2 - q^2) = x - y$.

Solution: The given equation can be written as:

$$\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2 = x - y$$

Taking $\sqrt{z}dz = dZ$, i.e., $Z = \frac{2}{3}z^{3/2}$, we get

$$\left(\frac{\partial Z}{\partial x}\right)^2 - \left(\frac{\partial Z}{\partial y}\right)^2 = x - y$$

$=> P^2 - Q^2 = x - y$, where $P = \frac{\partial Z}{\partial x}, Q = \frac{\partial Z}{\partial y}$

$=> P^2 - x = Q^2 - y$

Let $P^2 - x = Q^2 - y = a$ that

$P = \sqrt{a + x}, Q = \sqrt{a + y}$

Then the complete solution is:

$$z = \int \sqrt{a + x}dx + \int \sqrt{y + a}dy + b$$

$$z = \frac{(a+x)^{3/2}}{3/2} + \frac{(y+a)^{3/2}}{3/2} + b$$

or $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + c$

(7) Solve $z = px + qy + p^2q^2$.

Putting $p = a, q = b$ in the given equation, we get the complete solution.

i.e., $z = ax + by + a^2b^2$

(8) Solve $z^2 = pqxy$.

Solution: Let Charpit's auxiliary equations,

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{pqy-2yz} = \frac{dq}{pqx-2qz} \quad \ldots(1)$$

These give $\frac{xdp+pdx}{-2pxz} = \frac{ydq+qdy}{-2qyz}$

or $\frac{d(p/x)}{px} = \frac{d(qy)}{qy}$

Integrating, we get

$\log px = \log qy + \log b^2$

$=> px = qy(b^2) \quad \ldots(2)$

Solving the equations (1) and (2) for p and q, we get

$p = \frac{bz}{x}$ and $q = \frac{z}{by}$

Putting these values in $dz = pdx + qdy$, we get

$$dz = \frac{bz}{x}dx + \frac{z}{by}dy$$

or $\frac{dz}{z} = b\frac{dx}{x} + \frac{1}{b}\frac{dy}{y}$

Integrating, we get

$\log z = b\log x + \frac{1}{b}\log y + \log a$

or $z = ax^b y^{1/b}$ is the required solution.

(9) Solve $z^2(p^2+q^2) = x^2+y^2$.

Solution: Given $z^2(p^2+q^2) = x^2+y^2$ ...(1)

Let $z = \frac{1}{2}z \Rightarrow dZ = zdz$

$\therefore\ z^2p^2 = (zp)^2 = \left(z\frac{\partial z}{\partial x}\right)^2 = \left(\frac{\partial Z}{\partial x}\right)^2 = P^2$

$z^2q^2 = (zq)^2 = \left(z\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial Z}{\partial y}\right)^2 = Q^2$

Substituting in (1), we get

$\Rightarrow P^2+Q^2 = x^2+y^2 \Rightarrow P^2-x^2 = a^2, Q^2-y^2 = a^2$

$\Rightarrow P = \sqrt{a^2+x^2}, Q = \sqrt{a^2+y^2}$

Now, $dz = Pdx + Qdy$

$dz = \left(\sqrt{a^2+x^2}\right)dx + \left(\sqrt{a^2+y^2}\right)dy$

Integrating, we get

$z = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y\sqrt{a^2+y^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{y}{a} + b$

$\therefore$ General solution is

$z^2 = x\sqrt{a^2+x^2} + a^2\sinh^{-1}\frac{x}{a} + y\sqrt{a^2+y^2} + a^2\sinh^{-1}\frac{y}{a} + b$

where a and b are arbitrary constants.

(10) Solve $r + s - 6t = \cos(2x+y)$.

Solution: Given partial differential equation is $\left({D_x}^2 + D_xD_y - 6D_y\right)^2 z = \cos(2x+y)$

The auxiliary equation is (by replacing D_x with $'m'$ and by with $'1'$)

$m^2 + m - 6 = 0$

$\Rightarrow m = 2, -3$

Complimentary function $z_c = \phi_1(y+2x) + \phi_2(y-3x)$

Particular integration $z_p = \frac{1}{{D_x}^2 + D_xD_y - 6{D_y}^2}\cos(2x+y)$

Replace ${D_x}^2$ by $-a^2$; ${D_y}^2$ by $-b^2, D_xD_y$ with $-ab$

Here, $a = 2, b = 1$

$z_p = \frac{1}{-4-2+6}\cos(2x+y)$

Rewriting $f(D_x, D_y)$ as $(D_x - 2D_y)(D_x + 3D_y)$

Let, $z_p = \frac{1}{(D_x+3D_y)(D_x-2D_y)}\cos(2x+y)$

$= \frac{1}{(D_x+3D_y)}\int \cos(2x+y)$

$= \frac{1}{D_x+3D_y}\int \cos c$ (let $2x+y = c$)

$$= \frac{1}{D_x+3D_y} \cos c \int 1 dx = \frac{1}{D_x+3D_y} x \cos c$$

$$z_p = \frac{1}{D_x+3D_y} x \cos(2x + y)$$

Let $y = c + 3x$

$=> z_p = \int x \cos(5x + c)\, dx$

$= \frac{x \sin(5x+c)}{5} - \int \frac{\sin(5x+c)}{5} dx$

$= \frac{x \sin(5x+c)}{5} + \frac{\cos(5x+c)}{25}$

$\therefore c = y - 3x$

$=> z_p = \frac{x \sin(2x+y)}{5} + \frac{\cos(2x+y)}{25}$